

Finding cyclic vectors

Constanze Liaw
(conni@math.tamu.edu)

Texas A&M University

March 7, 2011

Joint work with A. Poltoratski and E. Abakumov

Outline

Rank one perturbations

- Cyclic for a.e. parameter
- Cyclic for all but one parameter
- Functions of Payley-Wiener class

Anderson-type Hamiltonians

- Setting
- Discrete random Schrödinger operator
- Anderson localization

Main Idea

- Vector φ is cyclic for T , if

$$\mathcal{H} = \text{clos span}\{(T - \lambda\mathbf{I})^{-1}\varphi : \lambda \in \mathbb{C} \setminus \mathbb{R}\}$$

- Operator T is called cyclic, if there exists a cyclic vector
- We will discuss two cases in which finding cyclic vectors is important

Rank One Perturbations and Applications

- A self-adjoint operator on Hilbert space \mathcal{H} , some vector φ
- Here for simplicity A bounded and $\varphi \in \mathcal{H}$
- Consider family of rank one perturbations

$$A + \alpha(\cdot, \varphi)\varphi \text{ for } \alpha \in \mathbb{R}$$

- WLOG $\varphi \in \mathcal{H}$ cyclic for A , i.e. $\mathcal{H} = \overline{\text{span}\{A^n\varphi : n \in \mathbb{N}\}}$
- E.g. Schrödinger operators with L^2 potentials, Jacobi matrices

Reformulation using the spectral theorem

- Recall $A + \alpha(\cdot, \varphi)\varphi$ for $\alpha \in \mathbb{R}$
- $\mu := \mu^\varphi$ denotes the unique spectral measure of A wrt φ ,
i.e. $(A^n \varphi, \varphi) = \int t^n d\mu(t)$ for $n \in \mathbb{N}$
- There exists a unitary operator $U : \mathcal{H} \rightarrow L^2(\mu)$ such that
 $UA = M_t U$ and $U\varphi = \mathbf{1}$.
- $A + \alpha(\cdot, \varphi)\varphi$ on $\mathcal{H} \stackrel{U}{\sim} A_\alpha := M_t + \alpha(\cdot, \mathbf{1})\mathbf{1}$ on $L^2(\mu)$

- Consider self-adjoint $A_\alpha = M_t + \alpha(\cdot, \mathbf{1})_{L^2(\mu)} \mathbf{1}$ on $L^2(\mu)$
- Spectral representation $V_\alpha : L^2(\mu) \rightarrow L^2(\mu_\alpha)$, i.e. unitary V_α such that $V_\alpha A_\alpha = M_s V_\alpha$ and $V_\alpha \mathbf{1} = \mathbf{1}$
- μ_α contains all spectral information of A_α
- For $f \in L^2(\mu)$ define $f_\alpha = V_\alpha f \in L^2(\mu_\alpha)$

Theorem

Assume the above setting. Let $0 \neq f \in L^2(\mu)$. Then we have:

- 1) The function f_α is a cyclic vector for the restriction $(A_\alpha)_{ac}$ for all but a countable number of $\alpha \in \mathbb{R}$.
- 2) The function f_α is a cyclic vector for $(A_\alpha)_s$ for Lebesgue a.e. $\alpha \in \mathbb{R}$.

- Different conclusions for the singular and the a.c. parts
- For mixed spectra there is an example: Function f_α non-cyclic for $(A_\alpha)_s$ for uncountably many α

Proof: 2) f_α is cyclic for $(A_\alpha)_s$ for a.e. $\alpha \in \mathbb{R}$

- Let μ_α denote the spectrum of $(A_\alpha)_s$
- Take $0 \neq f \in L^2(\mu)$
- By Poltoratski's Theorem and the classical fact that $(\mu_\alpha)_s \perp (\mu_\beta)_s$, it is possible to identify a unique function $\tilde{f} \in \mathbb{H}^\infty(\mathbb{C}_+)$ which has non-tangential boundary values f_α a.e. μ_α for all α
- Let $\Sigma = \{x \in \mathbb{R} : \tilde{f}(x) = 0\}$ denote the set of its zeros
- Assume that $\mu_\alpha(\Sigma) > 0$ for all $\alpha \in S$ for some $|S| > 0$
- Then by Aleksandrov's averaging theorem

$$\int_{\Sigma} dx = \int \mu_\alpha(\Sigma) d\alpha > 0$$

- Since \tilde{f} is analytic, this implies that $f \equiv 0$ (Contradiction)

Hermitian elements and cyclic vectors

- Consider unitary U on \mathcal{H} and a vector $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$
- Let $(\operatorname{Re} U)\varphi$ denote the space of linear combinations of $(U + U^*)^n \varphi$, $n \in \mathbb{N}$ with real coefficients
- $f \in \mathcal{H}$ is hermitian wrt U and φ , if $f \in (\operatorname{Re} U)\varphi$ and $f \perp \varphi$
- Consider the family of unitary rank one perturbations

$$U_\gamma = U + (\gamma - 1)(\cdot, U^{-1}\varphi)_{\mathcal{H}}\varphi \quad \gamma \in \mathbb{T}$$

- WLOG assume φ cyclic for U , i.e. $\overline{\operatorname{span}\{U^k \varphi : k \in \mathbb{N}\}^{\mathcal{H}}} = \mathcal{H}$

Theorem

In the above setting assume that $\sigma(U) = \sigma_s(U)$. Let $0 \neq f \in \mathcal{H}$ be hermitian with respect to U and φ . Fix a constant $c \in \mathbb{C} \setminus \{0\}$. Then the function $f - cb$ is cyclic for U_γ for all $\gamma \in \mathbb{T} \setminus \{e^{2i \arg c}\}$.

Special case of Clark Theory

Corollary

Let $\theta = z^n$ and $0 \neq p \in K_\theta = H^2 \ominus \theta H^2$ such that $p = \theta \bar{p}$ and $p(0) \neq 0$. Fix $c \in \mathbb{C} \setminus \{0\}$. Then

$$\{z : p(z) = c\} \subset \{e^{i2(\arg c + k\pi)/n} : k = 0, 1, \dots, n-1\}.$$

- The polynomials p have a very special structure:
 $\sum_{m=0}^{n-1} a_m z^m$, $a_m \in \mathbb{C}$, where $a_m = \overline{a_{n-1-m}}$
- Level sets are in an arithmetic progression (on the circle)

- Class Payley–Wiener functions PW_a equals $e^{-iaz} K_{e^{2iaz}}$
- So the cyclicity result for rank one perturbations is directly connected to the possible distributions of level sets of entire functions
- The uniqueness set Λ of functions of Payley–Wiener class has a characterization in terms of the question of completeness of $\{e^{i\lambda t} : \lambda \in \Lambda\}$ in the space $L^2(-a, a)$
- Goal: Analog to Euler's Formula $e^{i\varphi} = \cos \varphi + i \sin \varphi$ for entire functions from the model space $K_{e^{2iaz}}$

- separable Hilbert space \mathcal{H}
- sequence $\{\varphi_n\} \subset \mathcal{H}$
- $\omega = (\omega_1, \omega_2, \dots)$ distribution given by a probability measure \mathbb{P} on \mathbb{R}^∞ which satisfies Kolmogorov's 0-1 law
- Anderson-type Hamiltonian

$$H_\omega = H + \sum_n \omega_n(\cdot, \varphi_n) \varphi_n$$

- Assume that H_ω is almost surely essentially self-adjoint

- Special case: discrete random Schrödinger operator on $l^2(\mathbb{Z}^d)$

$$Hf(x) = -\Delta f(x) = - \sum_{n \in \mathbb{Z}^d, |n|=1} (f(x+n) - f(x)),$$

$$\varphi_n(x) = \delta_n(x) = \begin{cases} 1 & x = n, \\ 0 & \text{else} \end{cases}$$

- If moreover $d = 2$, then

(free Jacobi matrix + random entries on diagonal)

Recall the definition of the discrete random Schrödinger operator

$$H_\omega f(x) = - \sum_{n \in \mathbb{Z}^d, |n|=1} (f(x+n) - f(x)) + \sum_n \omega_n(\cdot, \delta_n) \delta_n.$$

Theorem

Assume that $\mathbb{P} = \prod_k \mu_k$ is a product measure of purely absolutely continuous measures μ_k on \mathbb{R} , and that the discrete random Schrödinger operator H_ω is cyclic almost surely. Then every non-zero vector $\psi \in \mathcal{H}$ is cyclic for H_ω almost surely.

- B. Simon (1994): For dense pure point spectrum on an interval, all δ_n are cyclic
- Jaksic–Last (2006): Certain cyclic vectors for $(H_\omega)_s$

Idea of the proof (Notions subcyclicity, non-orthog. graph)

- Introduce the notions of subcyclicity and the graph $\Gamma_T(\varphi_1, \varphi_2, \dots)$ of vectors $\varphi_1, \varphi_2, \dots$ under an operator T
- Subcyclicity is weaker than cyclicity
- For an operator T on a Hilbert space \mathcal{H} and vectors $\varphi_1, \varphi_2, \dots \in \mathcal{H}$ we define $\Gamma_T(\varphi_1, \varphi_2, \dots)$ to be the graph obtained by taking the vectors $\varphi_1, \varphi_2, \dots$ as vertices and connecting φ_k and φ_l exactly if

$$\text{Cycl}_T(\varphi_k) \not\subseteq \text{Cycl}_T(\varphi_l)$$

- If $T = H_\omega$, then the existence of edges is deterministic
- Let N invariant subspace and vector $\psi \in N$ be non-zero. The graph $\Gamma_{H_\omega}(\psi, \varphi_{n_1}, \varphi_{n_2}, \dots)$ is connected almost surely if and only if the vector ψ is subcyclic in N almost surely

Anderson localization

- In 1958 P. W. Anderson suggested that sufficiently large impurities in a semi-conductor could lead to spatial localization of electrons
- Here (localization=purely singular spectrum)
- For the discrete random Schrödinger operator, we consider $\mathbb{P} = \prod_k dx_{[-C,C]}$
- In dimension 1, Anderson localization takes place (for all $C > 0$)
- In dimensions 3 and higher, we have purely singular spectrum for C above a certain threshold
- Whether or not the model for $d \geq 2$ exhibits Anderson localization has been an open problem for over 50 years

- $H_\omega f(x) = - \sum_{n \in \mathbb{Z}^d, |n|=1} (f(x+n) - f(x)) + \sum_n \omega_n(\cdot, \delta_n) \delta_n$
- Then $(H_\omega)_s$ is cyclic almost surely

Corollary

In this setting, any non-zero vector is cyclic for $(H_\omega)_s$ almost surely.

- Anderson localization would be disproven (i.e. $(H_\omega)_{ac} \neq \mathbb{O}$), if one can find some (non-zero) vector which is non-cyclic almost surely for the random Schrödinger operator
- A computer experiment may be useful

Summary and open questions

Rank one perturbations

Cyclic for a.e. parameter

Cyclic for all but one parameter

Functions of Payley-Wiener class

Anderson-type Hamiltonians

Setting

Discrete random Schrödinger operator

Anderson localization