Deterministic spectral properties of Anderson-type Hamiltonians

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Outline

Setting

- separable Hilbert space $\mathcal{H}$
- complete sequence $\{\varphi_n\} \subset \mathcal{H}$
- $\omega = (\omega_1, \omega_2, \dots)$ distribution given by a probability measure $\mathbb{P}$ on $\mathbb{R}^\infty$ which satisfies Kolmogorov's 0-1 law
- General Anderson-type Hamiltonian

$$A_\omega = A + \sum_n \omega_n(\,\cdot\,, \varphi_n)\varphi_n$$

- Assume that A_ω is almost surely essentially self-adjoint
- Perturbation is almost surely a non-compact operator

Related

- Special case: discrete random Schrödinger operator on $l^2(\mathbb{Z}^d)$

$$Af(x) = -\triangle f(x) = -\sum_{n\in\mathbb{Z}^d, |n|=1} (f(x+n) - f(x)),$$

$$\varphi_n(x) = \delta_n(x) = \begin{cases} 1 & x = n, \\ 0 & \text{else} \end{cases}$$

- Jaksic–Last 2000, 2006: Provide good picture of the spectral properties, if $\{\varphi_n\}$ form orthonormal basis
- Main advantage of allowing non-orthogonal sequences of vectors is that the general Anderson-type Hamiltonians are connected to non-hermitian random matrices, while the assumption that $\{\varphi_n\}$ forms an orthonormal basis restricts to hermitian ones

Notation

- $T_{\mathrm{ac}} \sim \left(M_z \big|_{\oplus \int \mathcal{H}(z) d\mu_{\mathrm{ac}}(z)} \right)$
- $\sigma_{\mathrm{ess}}(T) = \sigma(T) \setminus \{$isolated point spectrum of finite mult.$\}$
- $A \sim B (\mathrm{mod}$ Class $X)$, if $(UAU^{-1} - B) \in$ Class X

General Perturbation Theory

- Weyl–vonNeuman: $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$ iff $A \sim B (\bmod \text{compact operators})$
- Kato–Rosenblum (Carey–Pincus): $A \sim B (\bmod \text{trace class})$. Then $A_{\text{ac}} \sim B_{\text{ac}}$
- Kolmogorov's 0-1 law and Anderson-type Hamiltonians: If the probability distribution $\mathbb{P}$ satisfies the 0-1 law, then those spectral properties that are invariant under finite rank perturbations are enjoyed by A_ω almost surely or almost never

Cauchy Transform

- Cauchy transform

$$K\sigma(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\sigma(t)}{t-z}\,, \qquad z \in \mathbb{C}_+$$

- $d\sigma_{\mathrm{ac}}(x) = \lim_{y\downarrow 0} \Im\, K\sigma(x+iy)\, dx$, $x \in \mathbb{R}$
- First moment of the Cauchy transform

$$K_1\sigma(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{t-z} - \frac{t}{t^2+1}\, d\sigma(t), \qquad z \in \mathbb{C}_+$$

- Locally $K_1\sigma$ and $K\sigma$ behave alike
- $K_1\sigma$ converges for any L^∞ function

Krein–Lifschits Spectral Shift of Rank One Perturbations

- For every $\alpha \in \mathbb{R}$ we can find $u \in L^\infty(\mathbb{R})$ with $-\pi < u \leq \pi$ and a constant $c \in \mathbb{R}$ such that

$$1 + \pi\alpha K\mu = e^{K_1 u + c} = (1 - \pi\alpha K\mu_\alpha)^{-1}$$

  where μ_α is the spectral measure of $A_\alpha = A + \alpha(\,\cdot\,, \varphi)\varphi$ wrt φ
- WLOG $\alpha > 0$, define u via the principal argument

$$u = \text{arg}(1 + \pi\alpha K\mu) = -\text{arg}(1 - \pi\alpha K\mu) \qquad \in [0, \pi]$$

- u jumps from 0 to π at singular points of μ
- Vice versa: Any function $u \in L^\infty(\mathbb{R})$ with $0 \leq u \leq \pi$ is the Krein spectral shift of the rank one perturbation $M_\mu + \alpha(\cdot, \mathbf{1})\mathbf{1}$ for any $\alpha > 0$

Theorem

- separable Hilbert space $\mathcal{H}$, complete sequence $\{\varphi_n\} \subset \mathcal{H}$, $\omega = (\omega_1, \omega_2, \ldots)$ distribution given by a probability measure $\mathbb{P}$ on $\mathbb{R}^\infty$ which satisfies Kolmogorov's 0-1 law
- Anderson-type Hamiltonian $A_\omega = A + \sum_n \omega_n(\,\cdot\,, \varphi_n)\varphi_n$

Theorem

We have almost surely $(\omega, \eta) \in \mathbb{P} \times \mathbb{P}$:

1) $(A_\omega)_{\mathrm{ac}} \sim (A_\eta)_{\mathrm{ac}}$,
2) $A_\omega \sim A_\eta(\mathrm{mod}$ *compact operator*$)$ *and*
3) *If* $(A_\omega)_{\mathrm{ess}}$ *is cyclic* $\omega \in \mathbb{P}$ *almost surely, then* $(A_\omega)_{\mathrm{ess}} \sim (A_\eta)_{\mathrm{ess}}(\mathrm{mod}$ *rank one*$)$.

Idea

- μ and ν denote the spectra of $(A_\omega)_{\text{ess}}$ and $(A_\eta)_{\text{ess}}$, respectively
- Produce $\{u_n^*\}$ which converges in measure. The pair of spectral measures corresponding to this limit function is then proven to be equivalent to the pair (μ, ν)
- Sufficies to construct u_n^*'s on $S = \operatorname{supp}_{\text{ess}} \mu_{\text{ac}} \backslash \partial \operatorname{supp}_{\text{ess}} \mu_{\text{ac}}$, because...(next slide)

Proof: 'Sufficies to construct $\{u_n^*\}$ on S'

- ...on $\mathbb{R}\backslash S = \operatorname{clos}(\sigma_{\text{ess}}(A_\omega)\backslash \operatorname{supp}_{\text{ess}} \mu_{\text{ac}})$ we can apply

Theorem (Poltoratski 2000)

Let A and B be two cyclic self-adjoint completely non-equivalent operators with purely singular spectrum satisfying $\sigma(A) = \sigma(B) = K \subset \mathbb{R}$, and $\sigma_{\text{pp}}(A) \cap \partial K = \sigma_{\text{pp}}(B) \cap \partial K = \varnothing$. Then we have $A \sim B (\operatorname{mod}$ rank one$)$

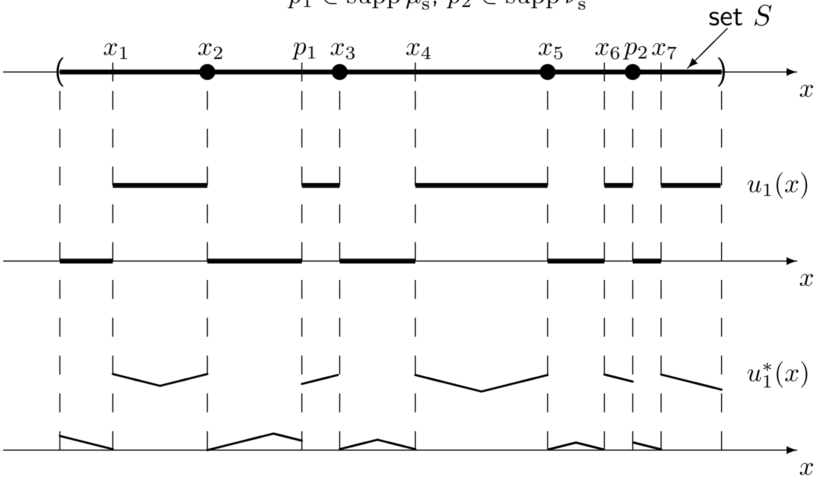
- Two operators A and B are completely non-equivalent, if and only if their spectral measures are mutually singular
- Corollary to Aronszajn–Donoghue: $A_\alpha = A + \alpha(\cdot, \varphi)\varphi$. Then $(\mu_\alpha)_{\text{s}} \perp (\mu_\beta)_{\text{s}}$ whenever $\alpha \neq \beta$

Definition of M_n and N_n

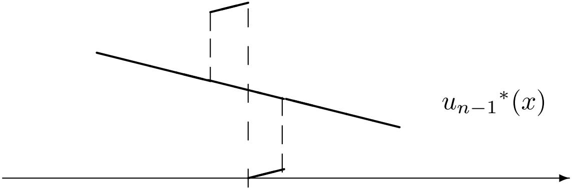
- Take sequences $\{M_n\}$ and $\{N_n\}$ of closed sets such that $|M_n| = |N_n| = 0$, $\mu_s(\mathbb{R}\backslash M_n) < 1/n$, $\nu_s(\mathbb{R}\backslash N_n) < 1/n$,...
- First we define u_n^*'s on S recursively
- Then show the appropriate equivalency of measures

Example of u_1^*

$$x_1, x_4, x_6, x_7 \in M_1;\ x_2, x_3, x_5 \in N_1$$
$$p_1 \in \operatorname{supp} \mu_s';\ p_2 \in \operatorname{supp} \nu_s'$$



Example: Inserting narrow 'slice' near $N_n \backslash N_{n-1}$



- WLOG assume $|\{u_n^* \neq u_{n-1}^*\}| \leq n^{-1}2^{-n}$ and

$$\int_{\{u_n^* \neq u_{n-1}^*\}} \frac{dx}{|x-t|} < 2^{-n} \qquad \text{for all } t \in M_{n-1} \cup N_{n-1}$$

- In measure $u_n^* \overset{n\to\infty}{\longrightarrow} u_0$, and u_0 yields measures μ_0 and ν_0 from the same family of rank one perturbations
- Remains to prove the equivalencies $\mu \sim \mu_0$ and $\nu \sim \nu_0$

$\mu_{\mathrm{ac}} \sim (\mu_0)_{\mathrm{ac}}$ and $\nu_{\mathrm{ac}} \sim (\nu_0)_{\mathrm{ac}}$

- Since $u_n^* \stackrel{n\to\infty}{\longrightarrow} u_0$ in measure and $|\{u_n^* = 0\} \cap \operatorname{supp}_{\mathrm{ess}} \mu_{\mathrm{ac}}| = 0$ for all n, it follows that $|\{u_0 = 0\} \cap \operatorname{supp}_{\mathrm{ess}} \mu_{\mathrm{ac}}| = 0$
- Similarly $|\{u_0 = \pi\} \cap \operatorname{supp}_{\mathrm{ess}} \mu_{\mathrm{ac}}| = 0$
- So $\Im K\sigma(x+iy) > 0$ a.e. $x \in \operatorname{supp}_{\mathrm{ess}} \mu_{\mathrm{ac}}$
- Trivially $\Im K\sigma(x+iy) = 0$ a.e. $x \notin \operatorname{supp}_{\mathrm{ess}} \mu_{\mathrm{ac}}$
- With $d\sigma_{\mathrm{ac}}(x) = \lim_{y\downarrow 0} \Im K\sigma(x+iy)\,dx$ we have

$$\mu_{\mathrm{ac}} \sim (\mu_0)_{\mathrm{ac}}$$

- An analog argument yields $\nu_{\mathrm{ac}} \sim (\nu_0)_{\mathrm{ac}}$

$\mu_s \sim (\mu_n^*)_s$ and $\nu_s \sim (\nu_n^*)_s$

- u_n and u_n^* are Lipschitz in neighborhood of M_n and

$$|K_1(u_n - u_n^*)| < \infty \qquad \text{on } M_n$$

- With $1 + \pi\alpha K\mu = e^{K_1 u + c}$ it follows that

$$\frac{K\mu_n}{K\mu_n^*} \sim e^{K_1(u_n - u_n^*)}$$

  in a neighborhood of M_n

- Since $\mu \sim \mu_n$ on M_n, we have $\mu \sim \mu_n^*$ on M_n
- Similarly $\nu_s \sim (\nu_n^*)_s$

$\mu_{\mathrm{s}} \sim (\mu_0)_{\mathrm{s}}$ and $\nu_{\mathrm{s}} \sim (\nu_0)_{\mathrm{s}}$

- $*$-weak convergence of $u_n^* dx$ and pointwise equality $1 + \pi\alpha K\mu = e^{K_1 u + c}$ imply $*$-weak convergence of $\{\mu_n^*\}$
- Hence $\mu_0 << \mu$ (on M_n)
- $1 + \pi\alpha K\mu = e^{K_1 u + c}$ and $\int_{\{u_n^* \neq u_{n-1}^*\}} \frac{dx}{|x-t|} < 2^{-n}$ for all $t \in M_{n-1} \cup N_{n-1}$ imply

$$1 - C2^{-n} < \frac{d\mu_n^*}{d\mu_{n-1}^*}(t) < 1 + C2^{-n}$$

  for all $t \in M_{n-1} \cup N_{n-1}$ and sufficiently large n
- So $\int_{X \cap L \cap M_n} \frac{d\mu_k^*(t)}{t^2+1} \stackrel{k\to\infty}{\longrightarrow} \delta > 0$ and with the $*$-weak convergence one can show that $\mu << \mu_0$ (on M_n)