Third Midterm Examination, Math 172 Spring 2010, Wilkerson Section.
April 26, 2010, 1:50 PM - 60 minutes- Blocker 160
No notes, books, calculators, tape players, earphones, etc. Show all work. Use back of pages for scratch.

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1. For each of these infinite series, determine convergence or divergence. Explain your answer. Your explanation must include the name of the test used to determine convergence or divergence.

(8+8+8 = 24 points)

a) \[ \sum_{n=0}^{\infty} n e^{-n} \]
   - Use ratio test \[ \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)e^{n-1}}{n e^n} \]
   - \[ \frac{(1+\frac{1}{n})}{e} \rightarrow \frac{1}{e} < 1 \]
   - Converges

b) \[ \sum_{n=5}^{\infty} \frac{n}{(n^2 - 1)} \]
   - By integral test \[ \int_{5}^{\infty} \frac{1}{(n^2 - 1)} \, dn \]
   - \[ \frac{n}{n^2 - 1} > \frac{1}{n} \]
   - So \[ \sum \frac{n}{n^2 - 1} \] diverges by comparison test for \( p = 1 \)

(c) \[ \sum_{n=3}^{\infty} \frac{n}{(n^3 + n^2 + n + 1)} \]
   - \[ \frac{1}{n^2} \text{ converges by p-test for } p = 2 \]
   - \[ \frac{1}{n^2} \text{ converges by p-test for } p = 2 \]
   - So \[ \sum \frac{n}{n^3 + n^2 + n + 1} \] converges by comparison test.
2. (20 points) For each of these power series, determine the radius of convergence using the ratio test.

a) \[
\sum_{n=1}^{\infty} \frac{n^2(x-5)^{2n}/4^n}{n^2 (x-5)^{2n}/4^n} = \frac{\sum_{n=1}^{\infty} \frac{(n+1)^2 (x-5)^{2n+2}}{4^{n+1}}}{\sum_{n=1}^{\infty} \frac{n^2 (x-5)^{2n}}{4^n}} = \frac{\sum_{n=1}^{\infty} \frac{(n+1)^2 (x-5)^{2n+2}}{4^{n+1}}}{\sum_{n=1}^{\infty} \frac{n^2 (x-5)^{2n}}{4^n}} = \frac{\sum_{n=1}^{\infty} \frac{(n+1)^2 (x-5)^{2n+2}}{4^{n+1}}}{\sum_{n=1}^{\infty} \frac{n^2 (x-5)^{2n}}{4^n}}
\]

so need \( \frac{(x-5)^2}{4} < 1 \), or \( |x-5| < 2 \), \( R=2 \)

b) \[
\sum_{n=1}^{\infty} \frac{2^n x^n/n!}{n^2 (x-5)^{2n}/4^n} = \frac{\sum_{n=1}^{\infty} \frac{2^{n+1} x^{n+1}}{(n+1)!}}{\sum_{n=1}^{\infty} \frac{2^n x^n}{n!}} = \frac{\sum_{n=1}^{\infty} \frac{2^{n+1} x^{n+1}}{(n+1)!}}{\sum_{n=1}^{\infty} \frac{2^n x^n}{n!}}
\]

so \( |x| < \infty \), \( R=\infty \)

3. For each of these functions, write out the Taylor series about \( x = 0 \) and its radius of convergence. You don’t have to do this from scratch - memory is ok.(16 points total)

a) \( f(x) = \frac{1}{2-x} \) = \( \frac{1}{2} \frac{1}{1-x/2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(x/2)^n}{n!} \), \( \frac{1}{2} |x/2| < 1 \)

so \( \frac{1}{2} |x| < 2 \)

\( R=2 \)

b) \( g(x) = \sin(3x) \) = \( \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+1}}{(2n+1)!} \)

so \( \sin(3x) = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+1}}{(2n+1)!} \)

\( R=\infty \)
4. Let
\[ f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \]
for \(|x| < 1\). (10 + 10 = 20 points)

a) Calculate \( f'(x) \) as a power series. Compare this answer to that obtained from differentiating \( \frac{1}{1+x} \).

\[
f' = \left( \sum_{n=0}^{\infty} (-1)^n x^n \right)' = \sum_{n=0}^{\infty} (-1)^n n x^{n-1}
\]

\[
f' = \left( \frac{1}{1+x} \right)' = \left( (1+x)^{-1} \right)' = (-1)(1+x)^{-2} \cdot 1
\]

\[
= -\frac{1}{(1+x)^2}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n n x^{n-1}
\]

b) Compute
\[
\int_0^x f(t) \, dt
\]
in two ways - first as a power series and second by doing the integral directly.

\[
\int_0^x \frac{1}{1+t} \, dt = \log (1+x) \bigg|_0^x = \log (1+x) - \log (1) = \log (1+x)
\]

\[
\int_0^x \sum_{n=0}^{\infty} (-1)^n t^n \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \frac{t^{n+1}}{n+1} \bigg|_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}
\]

5. Let \( P(x) = 4 + 2x - 3x^2 + x^4 + x^5 \). (7 + 7 + 6 = 20 points)

a) Compute the Taylor polynomial \( T_2(x) \) of order 2 about \( x = 1 \).

\[ P(1) = 4 + 2 - 3 + 1 + 1 = 5 \]
\[ P'(1) = 2 - 6 + 4 + 4 = 2 \]
\[ P''(1) = -12 + 12 = 0 \]
\[ P''(x) = -6 + 12x^2 + 20x^3 \]
\[ P''(1) = -6 + 12 + 20 = 26 \]

\[ T_2(x) = 5 + 5(x-1) + \frac{26}{2} (x-1)^2 \]

\[ = 5 + 5(x-1) + 13(x-1)^2 \]
b) Use $T_2(x)$ to approximate the value of $P(0.9)$. Hint: $x - 1 = 0.9 - 1 = -0.1$.

\[ P(0.9) \approx T_2(0.9) \]
\[ = 5 + 5(-0.1) + \frac{26}{2}(-0.1)^2 \]
\[ = 4.5 + 1.3 = 5.83 \]

c) We had a bound $|R_2(x)| < M|x-1|^3 / 6$, where $M = \max_I |f'''(x)|$, for the interval $I = [0.8, 1]$. One can compute that $M < 50$ in this example. Use this to estimate the error in using $T_2(x)$ to approximate $P(0.9)$.

\[ |R_2(0.9)| < \frac{50}{6} (0.1)^3 = \frac{50}{6000} = \frac{100}{12000} = \frac{1}{120} \]

So if $M$ is constant even $5 < \frac{1}{120}$

\[ M = 24x + 60x^2 \]

\[ |P''(x)| < 84 \text{ on } I \]

So $M$ should have been 84

So \[ \frac{84}{6000} \] is true error

The best value for $M$ is 84 but if you used this $50$ given that $5$ is ok.