MATH 304
Linear Algebra

Lecture 13:
Review for Test 1.
Topics for Test 1

Part I: Elementary linear algebra (Leon 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.
- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.
- Matrix algebra. Inverse matrix.
- Determinants: explicit formulas for $2 \times 2$ and $3 \times 3$ matrices, row and column expansions, elementary row and column operations.
Topics for Test 1

Part II: Abstract linear algebra (Leon 3.1–3.4, 3.6)

• Vector spaces (vectors, matrices, polynomials, functional spaces).
• Subspaces. Nullspace, column space, and row space of a matrix.
• Span, spanning set. Linear independence.
• Basis and dimension.
• Rank and nullity of a matrix.
Sample problems for Test 1

**Problem 1 (15 pts.)** Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

**Problem 2 (25 pts.)** Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix $A$.
(ii) Find the inverse matrix $A^{-1}$.
Problem 3 (20 pts.) Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set $S_1$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set $S_2$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.
(iii) The set $S_3$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.
(iv) The set $S_4$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

Problem 4 (30 pts.) Let

$$
B = \begin{pmatrix}
0 & -1 & 4 & 1 \\
1 & 1 & 2 & -1 \\
-3 & 0 & -1 & 0 \\
2 & -1 & 0 & 1
\end{pmatrix}.
$$

(i) Find the rank and the nullity of the matrix $B$.
(ii) Find a basis for the row space of $B$, then extend this basis to a basis for $\mathbb{R}^4$.
(iii) Find a basis for the nullspace of $B$. 

**Bonus Problem 5 (15 pts.)** Show that the functions \( f_1(x) = x, \ f_2(x) = xe^x, \) and \( f_3(x) = e^{-x} \) are linearly independent in the vector space \( C^\infty(\mathbb{R}) \).

**Bonus Problem 6 (15 pts.)** Let \( V \) be a finite-dimensional vector space and \( V_0 \) be a proper subspace of \( V \) (where proper means that \( V_0 \neq V \)). Prove that \( \dim V_0 < \dim V \).
Problem 1. Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

Let $p(x) = ax^2 + bx + c$. Then $p(1) = a + b + c$, $p(2) = 4a + 2b + c$, and $p(3) = 9a + 3b + c$. The coefficients $a$, $b$, and $c$ have to be chosen so that

$$\begin{cases}
a + b + c = 1, \\
4a + 2b + c = 3, \\
9a + 3b + c = 7.
\end{cases}$$

We solve this system of linear equations using elementary operations:

$$\begin{align*}
\begin{cases}
a + b + c = 1 \\
4a + 2b + c = 3
\end{cases} & \iff \\
\begin{cases}
a + b + c = 1 \\
3a + b = 2
\end{cases} \\
\begin{cases}
a + b + c = 1 \\
9a + 3b + c = 7
\end{cases} & \iff \\
\begin{cases}
3a + b = 2 \\
9a + 3b + c = 7
\end{cases}
\end{align*}$$
Thus the desired polynomial is \( p(x) = x^2 - x + 1 \).
Problem 2. Let \( A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix} \).

(i) Evaluate the determinant of the matrix \( A \).

Subtract the 4th row of \( A \) from the 3rd row:

\[
\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix}. 
\]
Expand the determinant by the 3rd row:

\[
\begin{vmatrix}
1 & -2 & 4 & 1 \\
2 & 3 & 2 & 0 \\
0 & 0 & -1 & 0 \\
2 & 0 & 0 & 1 \\
\end{vmatrix} = (-1) \begin{vmatrix}
1 & -2 & 1 \\
2 & 3 & 0 \\
2 & 0 & 1 \\
\end{vmatrix}.
\]

Expand the determinant by the 3rd column:

\[
(-1) \begin{vmatrix}
1 & -2 & 1 \\
2 & 3 & 0 \\
2 & 0 & 1 \\
\end{vmatrix} = (-1) \left( \begin{vmatrix}
2 & 3 \\
2 & 0 \\
\end{vmatrix} + \begin{vmatrix}
1 & -2 \\
2 & 3 \\
\end{vmatrix} \right) = -1.
\]
Problem 2. Let \( A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix} \).

(ii) Find the inverse matrix \( A^{-1} \).

First we merge the matrix \( A \) with the identity matrix into one \( 4 \times 8 \) matrix

\[
(A | I) = \begin{pmatrix} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.
Subtract 2 times the 1st row from the 2nd row:

\[
\begin{pmatrix}
1 & -2 & 4 & 1 & | & 1 & 0 & 0 & 0 \\
0 & 7 & -6 & -2 & | & -2 & 1 & 0 & 0 \\
2 & 0 & -1 & 1 & | & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 & | & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Subtract 2 times the 1st row from the 3rd row:

\[
\begin{pmatrix}
1 & -2 & 4 & 1 & | & 1 & 0 & 0 & 0 \\
0 & 7 & -6 & -2 & | & -2 & 1 & 0 & 0 \\
0 & 4 & -9 & -1 & | & -2 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 & | & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Subtract 2 times the 1st row from the 4th row:

\[
\begin{pmatrix}
1 & -2 & 4 & 1 & | & 1 & 0 & 0 & 0 \\
0 & 7 & -6 & -2 & | & -2 & 1 & 0 & 0 \\
0 & 4 & -9 & -1 & | & -2 & 0 & 1 & 0 \\
0 & 4 & -8 & -1 & | & -2 & 0 & 0 & 1
\end{pmatrix}
\]
Subtract 2 times the 4th row from the 2nd row:

$$
\begin{pmatrix}
1 & -2 & 4 & 1 & | & 1 & 0 & 0 & 0 \\
0 & -1 & 10 & 0 & | & 2 & 1 & 0 & -2 \\
0 & 4 & -9 & -1 & | & -2 & 0 & 1 & 0 \\
0 & 4 & -8 & -1 & | & -2 & 0 & 0 & 1 \\
\end{pmatrix}
$$

Subtract the 4th row from the 3rd row:

$$
\begin{pmatrix}
1 & -2 & 4 & 1 & | & 1 & 0 & 0 & 0 \\
0 & -1 & 10 & 0 & | & 2 & 1 & 0 & -2 \\
0 & 0 & -1 & 0 & | & 0 & 0 & 1 & -1 \\
0 & 4 & -8 & -1 & | & -2 & 0 & 0 & 1 \\
\end{pmatrix}
$$

Add 4 times the 2nd row to the 4th row:

$$
\begin{pmatrix}
1 & -2 & 4 & 1 & | & 1 & 0 & 0 & 0 \\
0 & -1 & 10 & 0 & | & 2 & 1 & 0 & -2 \\
0 & 0 & -1 & 0 & | & 0 & 0 & 1 & -1 \\
0 & 0 & 32 & -1 & | & 6 & 4 & 0 & -7 \\
\end{pmatrix}
$$
Add 32 times the 3rd row to the 4th row:

\[
\begin{pmatrix}
1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\
0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & -1 & 6 & 4 & 32 & -39
\end{pmatrix}
\]

Add 10 times the 3rd row to the 2nd row:

\[
\begin{pmatrix}
1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & -1 & 6 & 4 & 32 & -39
\end{pmatrix}
\]

Add the 4th row to the 1st row:

\[
\begin{pmatrix}
1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\
0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & -1 & 6 & 4 & 32 & -39
\end{pmatrix}
\]
Add 4 times the 3rd row to the 1st row:

$$
\begin{bmatrix}
1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\
0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \\
\end{bmatrix}
$$

Subtract 2 times the 2nd row from the 1st row:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\
0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \\
\end{bmatrix}
$$

Multiply the 2nd, the 3rd, and the 4th rows by $-1$:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\
0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \\
\end{bmatrix}
$$
Finally the left part of our $4 \times 8$ matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of $A$. Thus

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -6 & -4 & -32 & 39 \end{pmatrix}. $$
Problem 2. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix $A$.

*Alternative solution:* We have transformed $A$ into the identity matrix using elementary row operations. These included no row exchanges and three row multiplications, each time by $-1$.

It follows that $\det I = (-1)^3 \det A$.

$\implies \det A = -\det I = -1$. 
Problem 3. Determine which of the following subsets of \( \mathbb{R}^3 \) are subspaces. Briefly explain.

A subset of \( \mathbb{R}^3 \) is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(i) The set \( S_1 \) of vectors \((x, y, z) \in \mathbb{R}^3\) such that \(xyz = 0\).

\[(0, 0, 0) \in S_1 \implies S_1 \text{ is not empty}.
\]

\[xyz = 0 \implies (rx)(ry)(rz) = r^3xyz = 0.
\]

That is, \( \mathbf{v} = (x, y, z) \in S_1 \implies r\mathbf{v} = (rx, ry, rz) \in S_1.\)

Hence \( S_1 \) is closed under scalar multiplication.

However \( S_1 \) is not closed under addition.

Counterexample: \((1, 1, 0) + (0, 0, 1) = (1, 1, 1)\).
Problem 3. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(ii) The set $S_2$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.

$(0, 0, 0) \in S_2 \implies S_2$ is not empty.

$x + y + z = 0 \implies rx + ry + rz = r(x + y + z) = 0$. Hence $S_2$ is closed under scalar multiplication.

$x + y + z = x' + y' + z' = 0 \implies (x + x') + (y + y') + (z + z') = (x + y + z) + (x' + y' + z') = 0$.
That is, $v = (x, y, z), \ v' = (x', y', z') \in S_2 \implies v + v' = (x + x', y + y', z + z') \in S_2$.

Hence $S_2$ is closed under addition.
(iii) The set $S_3$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.

$y^2 + z^2 = 0 \iff y = z = 0$.

$S_3$ is a nonempty set closed under addition and scalar multiplication.

(iv) The set $S_4$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

$S_4$ is a nonempty set closed under scalar multiplication. However $S_4$ is not closed under addition.

Counterexample: $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$.
Problem 4. Let \( B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \).

(i) Find the rank and the nullity of the matrix \( B \).

The rank (= dimension of the row space) and the nullity (= dimension of the nullspace) of a matrix are preserved under elementary row operations. We apply such operations to convert the matrix \( B \) into row echelon form.

Interchange the 1st row with the 2nd row:

\[
\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}
\]
Add 3 times the 1st row to the 3rd row, then subtract 2 times the 1st row from the 4th row:

\[
\begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & -1 & 4 & 1 \\
0 & 3 & 5 & -3 \\
2 & -1 & 0 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & -1 & 4 & 1 \\
0 & 3 & 5 & -3 \\
2 & -1 & 0 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & -1 & 4 & 1 \\
0 & 3 & 5 & -3 \\
3 & -1 & 0 & 1
\end{pmatrix}
\]

Multiply the 2nd row by $-1$:

\[
\begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & -1 & 4 & 1 \\
0 & 3 & 5 & -3 \\
2 & -1 & 0 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & 1 & -4 & -1 \\
0 & 3 & 5 & -3 \\
0 & -3 & -4 & 3
\end{pmatrix}
\]

Add the 4th row to the 3rd row:

\[
\begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & 1 & -4 & -1 \\
0 & 3 & 5 & -3 \\
2 & -1 & 0 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & 1 & -4 & -1 \\
0 & 0 & 1 & 0 \\
0 & -3 & -4 & 3
\end{pmatrix}
\]
Add 3 times the 2nd row to the 4th row:

\[
\begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & 1 & -4 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & -16 & 0
\end{pmatrix}
\]

Add 16 times the 3rd row to the 4th row:

\[
\begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & 1 & -4 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Now that the matrix is in row echelon form, its rank equals the number of nonzero rows, which is 3. Since
\[
(\text{rank of } B) + (\text{nullity of } B) = (\text{the number of columns of } B) = 4,
\]
it follows that the nullity of \( B \) equals 1.
Problem 4. Let \( B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \).

(ii) Find a basis for the row space of \( B \), then extend this basis to a basis for \( \mathbb{R}^4 \).

The row space of a matrix is invariant under elementary row operations. Therefore the row space of the matrix \( B \) is the same as the row space of its row echelon form:

\[
\begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

The nonzero rows of the latter matrix are linearly independent so that they form a basis for its row space:
\[ \mathbf{v}_1 = (1, 1, 2, -1), \quad \mathbf{v}_2 = (0, 1, -4, -1), \quad \mathbf{v}_3 = (0, 0, 1, 0). \]

To extend the basis \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) to a basis for \( \mathbb{R}^4 \), we need a vector \( \mathbf{v}_4 \in \mathbb{R}^4 \) that is not a linear combination of \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \).

It is known that at least one of the vectors \( \mathbf{e}_1 = (1, 0, 0, 0), \mathbf{e}_2 = (0, 1, 0, 0), \mathbf{e}_3 = (0, 0, 1, 0), \) and \( \mathbf{e}_4 = (0, 0, 0, 1) \) can be chosen as \( \mathbf{v}_4 \).

In particular, the vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_4 \) form a basis for \( \mathbb{R}^4 \). This follows from the fact that the \( 4 \times 4 \) matrix whose rows are these vectors is not singular:

\[
\begin{vmatrix}
1 & 1 & 2 & -1 \\
0 & 1 & -4 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{vmatrix} = 1 \neq 0.
\]
Problem 4. Let \( B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \).

(iii) Find a basis for the nullspace of \( B \).

The nullspace of \( B \) is the solution set of the system of linear homogeneous equations with \( B \) as the coefficient matrix. To solve the system, we convert \( B \) to reduced row echelon form:

\[
\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

\( \implies x_1 = x_2 - x_4 \quad x_3 = 0 \)

General solution: \( (x_1, x_2, x_3, x_4) = (0, t, 0, t) = t(0, 1, 0, 1) \).

Thus the vector \( (0, 1, 0, 1) \) forms a basis for the nullspace of \( B \).
**Bonus Problem 5.** Show that the functions $f_1(x) = x$, $f_2(x) = x e^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

The functions $f_1, f_2, f_3$ are linearly independent whenever the Wronskian $W[f_1, f_2, f_3]$ is not identically zero.

$$W[f_1, f_2, f_3](x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix} = \begin{vmatrix} x & xe^x & e^{-x} \\ 1 & e^x + xe^x & -e^{-x} \\ 0 & 2e^x + xe^x & e^{-x} \end{vmatrix}$$

$$= e^{-x} \begin{vmatrix} x & xe^x & 1 \\ 1 & e^x + xe^x & -1 \\ 0 & 2e^x + xe^x & 1 \end{vmatrix} = x \begin{vmatrix} 1+x & -1 \\ 2+x & 1 \end{vmatrix} = x(2x+3) + 2 = 2x^2 + 3x + 2.$$
**Bonus Problem 5.** Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

*Alternative solution:* Suppose that $af_1(x) + bf_2(x) + cf_3(x) = 0$ for all $x \in \mathbb{R}$, where $a, b, c$ are constants. We have to show that $a = b = c = 0$.

Let us differentiate this identity:

$$ax + bxe^x + ce^{-x} = 0,$$

$$a + be^x + bxe^x - ce^{-x} = 0,$$

$$2be^x + bxe^x + ce^{-x} = 0,$$

$$3be^x + bxe^x - ce^{-x} = 0,$$

$$4be^x + bxe^x + ce^{-x} = 0.$$  

(\text{the 5th identity})−(\text{the 3rd identity}):  

\[ 2be^x = 0 \implies b = 0. \]

Substitute $b = 0$ in the 3rd identity:  

\[ ce^{-x} = 0 \implies c = 0. \]

Substitute $b = c = 0$ in the 2nd identity:  

\[ a = 0. \]
Bonus Problem 5. Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Alternative solution: Suppose that $ax + bxe^x + ce^{-x} = 0$ for all $x \in \mathbb{R}$, where $a, b, c$ are constants. We have to show that $a = b = c = 0$.

For any $x \neq 0$ divide both sides of the identity by $xe^x$:
$$ae^{-x} + b + cx^{-1}e^{-2x} = 0.$$  
The left-hand side approaches $b$ as $x \to +\infty$. $\implies b = 0$

Now $ax + ce^{-x} = 0$ for all $x \in \mathbb{R}$. For any $x \neq 0$ divide both sides of the identity by $x$:
$$a + cx^{-1}e^{-x} = 0.$$  
The left-hand side approaches $a$ as $x \to +\infty$. $\implies a = 0$

Now $ce^{-x} = 0 \implies c = 0$. 
**Bonus Problem 6.** Let $V$ be a finite-dimensional vector space and $V_0$ be a proper subspace of $V$ (where proper means that $V_0 \neq V$). Prove that $\dim V_0 < \dim V$.

Any vector space has a basis. Let $v_1, v_2, \ldots, v_k$ be a basis for $V_0$.

Vectors $v_1, v_2, \ldots, v_k$ are linearly independent in $V$ since they are linearly independent in $V_0$. Therefore we can extend this collection of vectors to a basis for $V$ by adding some vectors $w_1, \ldots, w_m$. As $V_0 \neq V$, we do need to add some vectors, i.e., $m \geq 1$.

Thus $\dim V_0 = k$ and $\dim V = k + m > k$. 