Parking Functions, Interpolation Polynomials, and the Operator Method

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Outline

The First Story

Part 1. Classical Parking Functions and Goncarov Polynomials

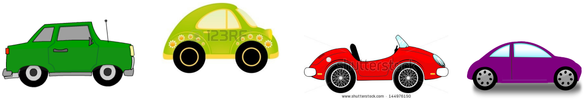
Original Definition

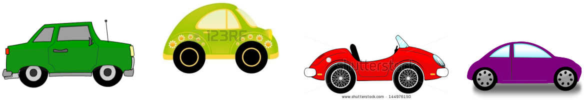
Definition (Konheim and Weiss 66)

Cars $C_1, \ldots, C_n$ want to park on a one-way street with ordered parking spaces $1, \ldots, n$. Each car C_i has a preferred space a_i.

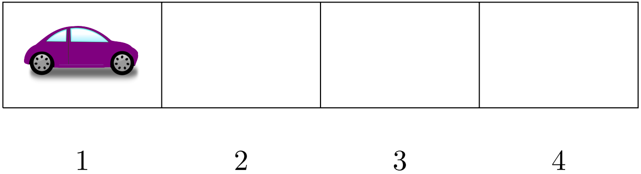
- *Each car goes to its preferred space and park there if the space is empty.*
- *Otherwise, it moves forward and parks in the next available space.*
- *If there is no space available, then the car leaves the street and fails to park.*

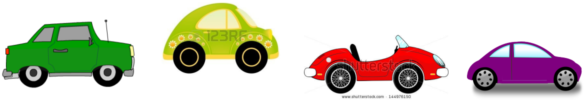
The sequence $(a_1, \ldots, a_n)$ is called a parking function of length n if all the cars can park.





With preference sequence $(1, 4, 2, 2)$, they park as:





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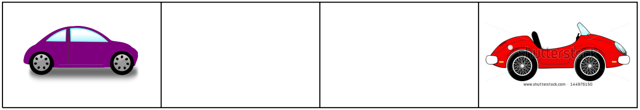
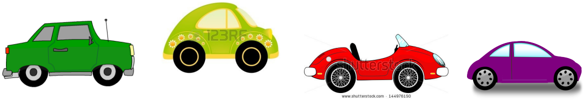


1 2 3 4



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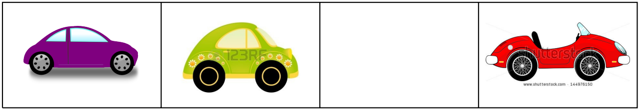

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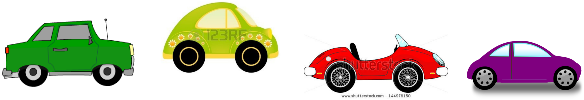

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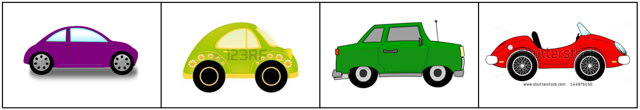
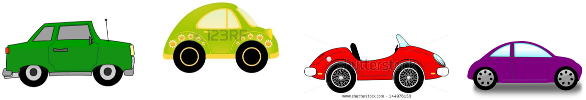


1	2	3	4



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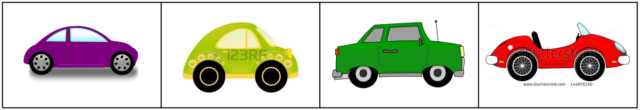


1 2 3 4

With preference sequence $(1, 4, 3, 3)$, the last car cannot park.

Easy: $(a_1, \ldots, a_n)$ is a parking function if and only if it has at least i terms less than or equal to i.
Formal 1: $(a_1, \ldots, a_n)$ is a parking function if and only if there is a permutation π such that $0 < a_i \leq \pi_i$.
Formal 2 The nondecreasing rearrangement
$a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$ satisfies $a_{(i)} \leq i$ for all i.

Corollary

Any permutation of a parking function is still a parking function.

Brief History

- 1966: (Konheim & Weiss) random hashing functions, $(n+1)^{n-1}$.
- 1970's: (Foata, Riordan, Knuth) labeled trees and parking functions
- 1990's: (Stanley) interval orders, hyperplane arrangements, noncrossing partitions, symmetric functions, stochastic processes, and associahedrons, ...
- 1998-: (Garsia, Haiman, Haglund, etc.) Diagonal harmonics, MacDonald polynomials, representation theory, Shuffle conjecture...
- 2000's: (Postnikov, Holtz,etc) commutative algebra, polynomial ideals, approximation theory, box spline, ...

Many Generalizations

- k-parking functions
- **u**-parking functions and parking polytope
- rational parking functions
- G-parking functions and sandpile model

More recent:

- Parking functions on trees and digraphs (King's talk)
- parking sequences with different car sizes
- parking sequences with multi-choices

Vector Parking Functions

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ be a sequence of non-decreasing positive integers.
For a sequence $(x_1, x_2, \ldots, x_n)$, the *order statistics* is the nondecreasing rearrangement $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$.

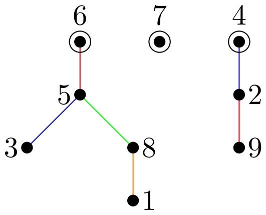
Definition

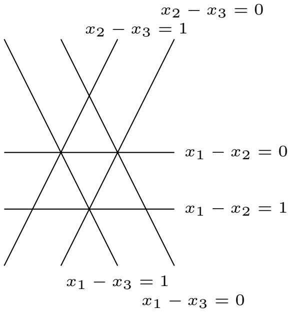
A $\mathbf{u}$*-parking function of length* n *is a sequence* $(x_1, x_2, \ldots, x_n)$ *of positive integers whose order statistics satisfy* $x_{(i)} \leq u_i$ *for all* i*.*

When $\mathbf{u} = (1, 2, \ldots, n)$, we obtain the classical case.

Combinatorial structures

Rooted forests with colored edges, extended Shi hyperplane arrangements, and k-divisible noncrossing partitions.





Counting u-Parking Functions

Let $PK_n(\mathbf{u})$ be the number of $\mathbf{u}$-parking functions of length n. For example, for $\mathbf{u} = (2, 5)$, $\mathbf{u}$-parking functions include

$$11, 12, 13, 14, 15, 22, 23, 24, 25$$

and their permutations.

$$PK_n((2, 5)) = 16$$

Proposition

$PK_n(\mathbf{u})$ is a polynomial of $u_1, u_2, \ldots, u_n$ with degree n.

Goncarov Polynomials

Definition (Goncarov Interpolation)

Given two sequences of real or complex numbers $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$, find a polynomial $p(x)$ of degree n such that for each $i, 0 \le i \le n$, the ith derivative $p^{(i)}(x)$ evaluated at a_i equals b_i.

For the case $b_i = n!\delta_{i,n}$, the unique solution of degree n is the Goncarov polynomial

$$g_n(x; a_0, a_1, \ldots, a_{n-1}).$$

Some special cases

$$g_0(x) = 1,$$

$$g_1(x; a_0) = x - a_0,$$

$$g_2(x; a_0, a_1) = x^2 - 2a_1 x + 2a_0 a_1 - a_0^2,$$

$$g_3(x; a_0, a_1, a_2) = x^3 - 3a_2 x^2 + 3(2a_1 a_2 - a_1^2)x - a_0^3 + 3a_0^2 a_2 - 6a_0 a_1 a_2 + 3a_0 a_1^2.$$

In addition,

$$g_n(x; a, a, \ldots, a) = (x - a)^n$$

$$g_n(x; a, a + b, a + 2b, \ldots, a + (n-1)b) = (x - a)(x - a - nb)^{n-1}.$$

Nice properties of Goncarov polynomials

- *A determinant formula* for $g_n(x; \mathbf{a})$
- *Linear recurrence*

$$x^n = \sum_{i=0}^{n} \binom{n}{i} a_i^{n-i} g_i(x; \mathbf{a}).$$

- *Appell relation.*

$$e^{xt} = \sum_{n=0}^{\infty} g_n(x; \mathbf{a}) \frac{t^n e^{a_n t}}{n!}.$$

- *Differential Relation.* $g_n(a_0) = 0$ and

$$Dg_n(x; a_0, a_1, \ldots, a_{n-1}) = ng_{n-1}(x; a_1, a_2, \ldots, a_{n-1}),$$

- *Shift-invariance formula*

$$g_n(x + \xi; a_0 + \xi, a_1 + \xi, \ldots, a_{n-1} + \xi) = g_n(x; \mathbf{a}).$$

- *Binomial relation*

$$g_n(x + y; \mathbf{a}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(y; a_i, \ldots, a_{n-1}) x^i.$$

In particular,

$$g_n(x; \mathbf{a}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(0, a_i, \ldots, a_{n-1}) x^i.$$

Theorem (Kung & Yan 03)

Let $\mathbf{u} = (u_1, u_2, \dots)$ *be a sequence of non-decreasing positive integers. Then we have*

$$\begin{aligned} PK_n(\mathbf{u}) &= g_n(x; x-u_1, x-u_2, \dots, x-u_n) \\ &= g_n(0; -u_1, -u_2, \dots, -u_n) \\ &= (-1)^n g_n(0; u_1, u_2, \dots, u_n). \end{aligned}$$

Obtained by a combinatorial interpretation of the linear recurrence formula.

Corollaries:

1. $P_n(1, 2, \ldots, n) = (n+1)^{n-1}$
2. $P_n(a, a+b, a+2b, \ldots, a+(n-1)b) = a(a+nb)^{n-1}$
3. $P_n(bu_1, bu_2, \ldots, bu_n) = b^n P_n(u_1, u_2, \ldots, u_n)$.
4. In general, $P_n(u_1, \ldots, u_n) = n! \det(M)$ where $M_{i,j} = u_i^{j-i+1}/(j-i+1)!$.

Applications and Extensions

Any algebraic formula of Goncarov polynomials gives a result on **u**-parking functions.

- [Knuth 98, Kung & Yan 03] Moments of sum of parking functions, both exact formulas and asymptotics
- [Pitman & Stanley 98] Parking polytope and empirical distribution.
- [Khare, Lorentz & Yan 14] Higher dimension interpolation and integer sequences. (Applications in epidemic model and insurance model)

A Side Story

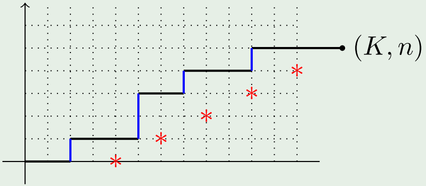
Part II. Lattice Path Counting and the Difference Operator

Lattice path in plane

Lattices paths from $(0,0)$ to (K,n) with steps $(1,0)$ or $(0,1)$ $= (x_0, \ldots, x_n)$ where (x_i, i) is the right-most point on the i-th row.

Example

Lattice path $(2,5,5,7,10)$ with right boundary $(4,6,8,10,12)$.



Lattice path with right boundary

For K sufficiently large, let $LP_n(a_0, \ldots, a_{n-1})$ be the number of lattice paths from $(0,0)$ to (K, n) such that $x_i < a_i$ for $0 \leq i \leq n-1$.

1. $LP_1(\mathbf{a}) = a_0$, $LP_2(\mathbf{a}) = a_0 a_1 - \frac{a_0(a_0-1)}{2}$.
2. For $\mathbf{a} = (1, 2, 3, \ldots, n)$, LP_n is the n-th Catalan number.
3. Reflection principle, penetrating analysis, recurrence, and generating function method

Proposition

$LP_n(\mathbf{a})$ is a polynomial of $a_0, a_1, \ldots, a_{n-1}$ with degree n.

Difference Goncarov polynomials

Theorem (Kung, Sun &Y'05)

$$\begin{aligned} LP_n(a_0, \ldots, a_{n-1}) &= \frac{1}{n!} \tilde{g}_n(x; x - a_0, \cdots, x - a_{n-1}) \\ &= \frac{1}{n!} \tilde{g}_n(0; -a_0, \ldots, -a_{n-1}). \end{aligned}$$

Difference Goncarov polynomials

Theorem (Kung, Sun &Y'05)

$$LP_n(a_0, \ldots, a_{n-1}) = \frac{1}{n!}\tilde{g}_n(x; x - a_0, \cdots, x - a_{n-1})$$
$$= \frac{1}{n!}\tilde{g}_n(0; -a_0, \ldots, -a_{n-1}).$$

Here $\tilde{g}_n(x; \mathbf{a})$ is the **difference analog of Goncarov polynomials** by replacing the derivative operator with **backward difference operator**.

Difference Operator

The backward difference operator

$$\Delta p(x) = p(x) - p(x-1),$$

with basis $x^{(n)} = x(x+1)\cdots(x+n-1)$, i.e., $\Delta x^{(n)} = nx^{(n-1)}$.

Definition

$\tilde{g}_n(x;\mathbf{a})$ is the unique polynomial of degree n such that for each i, $0 \le i \le n$, the ith iteration of Δ on $\tilde{g}$, evaluated at a_i, is $n!\delta_{n,i}$, i.e.,

$$\Delta^{(i)}\tilde{g}(x;\mathbf{a})|_{x=a_i} = n!\delta_{n,i}.$$

[De Mier, Kung & Yan 09] Algebraicity of the generating function of $LP_n(\mathbf{a})$ when $\mathbf{a}$ is periodic.

The Big Picture

**Part III. Interpolation Polynomials,
Operator Methods,
and Theory of Binomial Enumeration**

Question: What if the cars are coming in blocks with certain structures?

e.g.,

$$\begin{aligned} & \text{lattice path} \\ \leftrightarrow \quad & \text{cars are in groups with linear order} \\ & \text{each group has the same preference} \end{aligned}$$

The Bridge: Operator Method

- with Kahaner and Odlyzko: Finite Operator Calculus
- with Mullin: Foundations III: Theory of Binomial Enumeration.

Delta operators

$P[x]$: vector space of all polynomials in x over a field P.
Shift operator E_a from $P[x]$ to $P[x]$: $E_a(f(x)) = f(x+a)$.

Definition

A delta operator $\mathfrak{d}$ is a linear operator from $P[x]$ to $P[x]$ that is

1. *shift-invariant: $\mathfrak{d} \circ E_a = E_a \circ \mathfrak{d}$ for all a, and*
2. *satisfying $\mathfrak{d}(x) = a$ for some nonzero constant a.*

Delta operators are degree-reducing.

Basic polynomials

Definition

Let $\mathfrak{d}$ be a delta operator. A polynomial sequence $(p_n(x))_{n\geq 0}$ is called the sequence of basic polynomials of $\mathfrak{d}$ if

1. $p_0(x) = 1$;
2. $p_n(0) = 0$ *whenever* $n \geq 1$;
3. $\mathfrak{d}(p_n(x)) = np_{n-1}(x)$.

delta operator	basic polynomials
D	standard power x^n
$I - E_{-1}$	rising factorial $x(x+1)\cdots(x+n-1)$
$E_1 - I$	lower factorial $x(x-1)\cdots(x-n+1)$
$E_a - E_b$	Gould polynomial $x\prod_{i=1}^{n-1}(x-ia-(n-i)b)$
$E_a D$	Abel polynomial $x(x-na)^{n-1}$

Connections

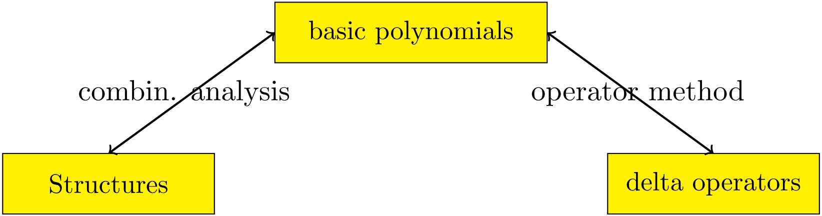
Theorem

Every delta operator has a unique sequence of basic polynomials, which is a sequence of binomial type, i.e., satisfies

$$p_n(x+y) = \sum_{k \geq 0} \binom{n}{k} p_k(x) p_{n-k}(y) \qquad \text{for all } n.$$

Conversely, every sequence of polynomials of binomial type is the basic sequence for some delta operator.

Big picture


basic polynomials
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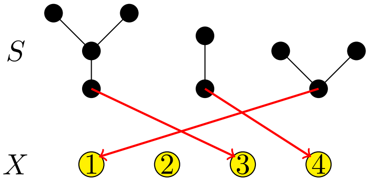
Reluctant Functions

Definition (Mullin & Rota)

A ***reluctant function*** *f from S to X is a function $f : S \to S \cup X$ in which for any $s \in S$, there is a positive integer k such that $f^{(k)}(s) \in X$. This $f^{(k)}(s)$ is the* ***final image*** *of s and is written as $f^{*}(s)$.*

Final image



If $S = \{s_1, s_2, \ldots, s_n\}$, then the *final image sequence* is

$$(f^*(s_1), f^*(s_2), \ldots, f^*(s_n)).$$

Binomial classes

Binomial classes of reluctant functions $\mathcal{B}$: a family of reluctant functions such that

1. Given S and X, $|F(S, X)|$ depends on the sizes of S and X,
2. Binomial property: there is a natural isomorphism

$$\mu : F(S, X \oplus Y) \to \dot{\bigcup_{A \subseteq S}} (F(A, X) \otimes F(S \setminus A, Y)).$$

Theorem

Let $p_n(x)$ be the size of $F(S, X)$ in a binomial class $\mathcal{B}$, where $n = |S|$ and $x = |X|$. Then $\{p_n(x) : n \geq 0\}$ is of binomial type.

Interpolation with delta operators

Generalized Goncarov Interpolation

Given two sequences $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ of real or complex numbers and a delta operator $\mathfrak{d}$, find a polynomial $p(x)$ of degree n such that

$$\varepsilon_{a_i} \mathfrak{d}^i(p(x)) = b_i \qquad \text{for each } i = 0, 1, \ldots, n.$$

For the case $b_i = n!\delta_{i,n}$, the unique solution of degree n is the delta Goncarov polynomial

$$t_n(x; \mathfrak{d}, a_0, a_1, \ldots, a_{n-1}).$$

Analytic Properties

$t_n(x; \mathfrak{d}, a_0, a_1, \dots, a_{n-1})$ enjoys similar properties as the classical Goncarov polynomial:

- determinant formula,
- combinatorial formula in terms on ordered partitions,
- linear expansion and recurrence,
- Appell-type generating function,
- shift-invariance formula,
- binomial expansion formula, etc.

Question: What role do these interpolation polynomials play in combinatorial structures?

Answer: They enumerate binomial structures with an order-statistic constraint.

Goncarov polynomials in Enumeration

For a binomial class $\mathcal{B}$, assume

- $p_n(x) = |F(S, X)|$ if $|S| = n$ and $|X| = x$,
- $\mathfrak{d}$ is the delta operator associated with $\{p_n(x) : n \geq 0\}$.
- $t_n(x; \mathfrak{d}, \mathbf{a})$ is the delta Goncarov polynomial.
- $\mathbf{u}$ is a sequence of nondecreasing positive integers.

Goncarov polynomials in Enumeration

For a binomial class $\mathcal{B}$, assume

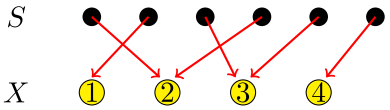
- $p_n(x) = |F(S,X)|$ if $|S| = n$ and $|X| = x$,
- $\mathfrak{d}$ is the delta operator associated with $\{p_n(x) : n \geq 0\}$.
- $t_n(x; \mathfrak{d}, \mathbf{a})$ is the delta Goncarov polynomial.
- $\mathbf{u}$ is a sequence of nondecreasing positive integers.

Theorem (Lorentz, Tringali, Yan)

The number of reluctant functions in $F(S,X)$ whose final images are $\mathbf{u}$-parking functions is given by

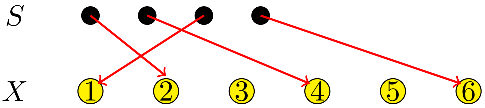
$$t_n(x; \mathfrak{d}, (x-u_i)_{i\geq 0}) = t_n(0; \mathfrak{d}, -\mathbf{u}).$$

Examples



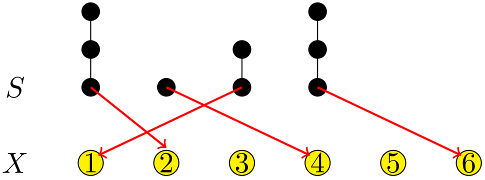
Example (1)

Binomial Class: functions from S to X,
Basic polynomials: $p_n(x) = x^n$
delta operator: derivative D
u-parking functions: classical Goncarov polynomials
number of parking functions: $(n+1)^{n-1}$.



Example (2)

Binomial Class: 1-1 functions from S to X,
Basic polynomials: $p_n(x) = x(x-1)\cdots(x-n+1)$
delta operator: forward difference operator $E_1 - I$
u-parking functions: $t_n(0; E_1 - I, -\mathbf{u})$
number of parking functions: $n!$.



Example (3)

Binomial Class: trees in S are labeled paths, 1-1 function
Basic polynomials: $p_n(x) = x(x+1)\cdots(x+n-1)$
delta operator: backward difference operator $I - E_{-1}$
u-parking functions: $t_n(0; I - E_{-1}, -\mathbf{u}) = \tilde{g}_n(0; -\mathbf{u})$
number of parking functions: $n!C_n$, where C_n is the Catalan number.

Example (4)

Binomial Class: trees in S are labeled paths, any function
Basic polynomials: $p_n(x) = L_n(-x)$ the Laguerre polynomial

$$L_n(-x) = \sum_k \frac{n!}{k!}\binom{n-1}{k-1} x^k.$$

delta operator: $K = D(D-I)^{-1}$
u-parking functions: $t_n(0; K, -\mathbf{u})$
number of parking functions: $\sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}(1+n)^{k-1}$.

Example (5)

Binomial Class: trees in S are monotone paths
Basic polynomials: $p_n(x) = \sum_{k=1} S(n,k)x^k$, the exponential polynomial.
delta operator: $\log(I + D)$
number of parking functions: $\sum_{k=1}^{n} S(n,k)(1+n)^{k-1}$, where $S(n,k)$ is the Stirling number of the second kind.

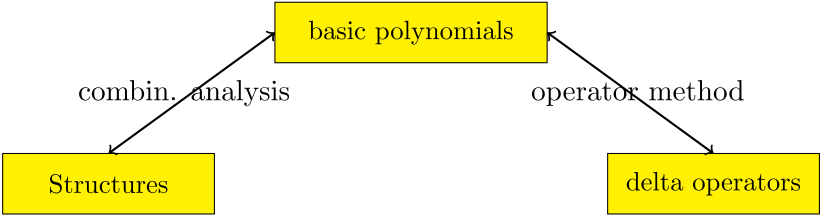
Example (6)

Binomial Class: trees in S are arbitrary
Basic polynomials: $p_n(x) = x(x+n)^{n-1}$
delta operator: $E_1 D$
number of parking functions: $(1+2n)^{n-1}$.

Example (7)

Binomial Class: trees in S are stars
Basic polynomials: $p_n(x) = \sum_{k\geq 0} \binom{n}{k} k^{n-k} x^k$, (inverse of the Abel polynomials).
delta operator: $g(D)$ where g is the compositional inverse of $f(t) = te^t$.
number of parking functions: $\sum_{k=1}^{n} \binom{n}{k} k^{n-k} (1+n)^{k-1}$.

Summary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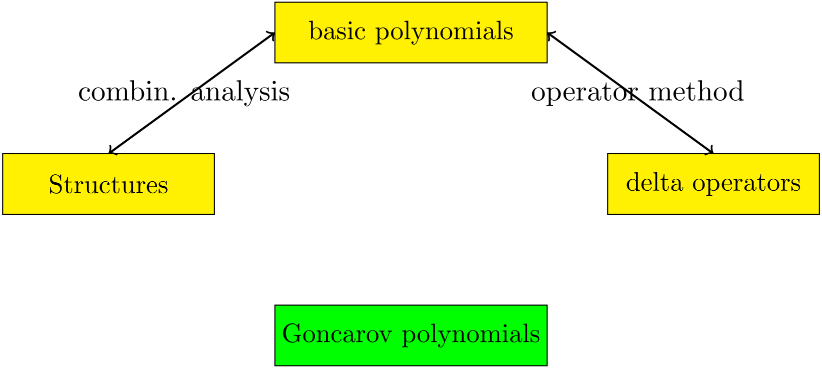
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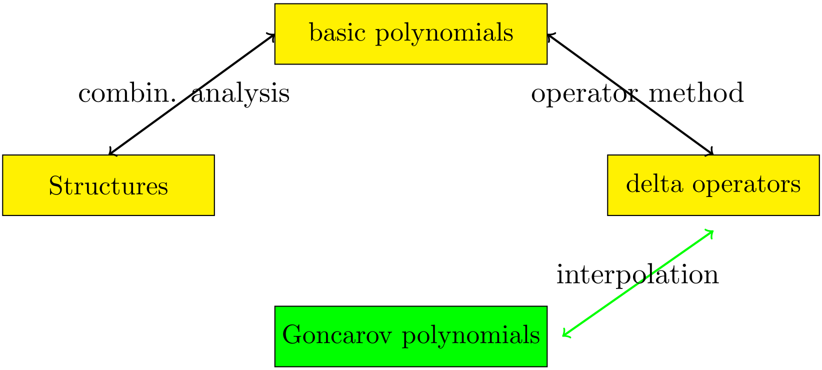
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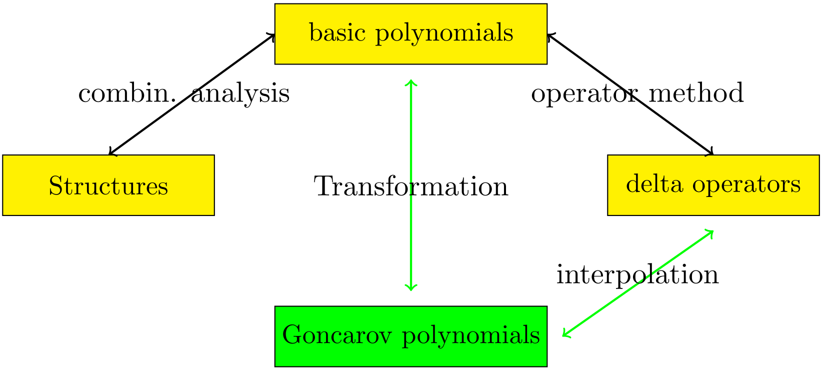
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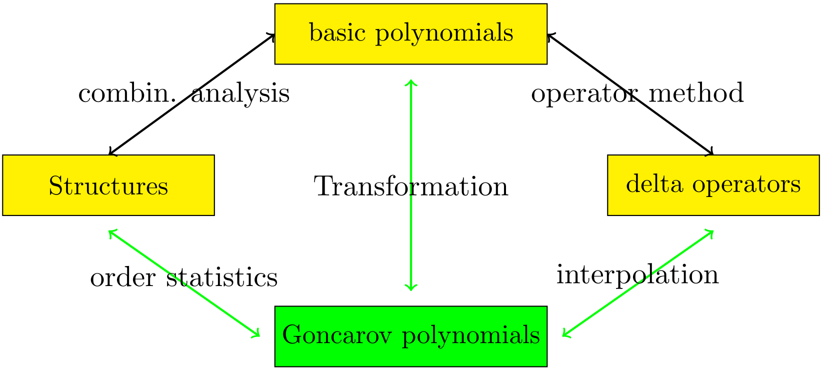
interpolation
Goncarov polynomials

Summary


basic polynomials
combin. analysis
operator method
Structures
Transformation
delta operators
order statistics
interpolation
Goncarov polynomials

Further Research

1. Goncarov polynomials on Exponential Families, (joint with Adeniran)
2. Basic polynomials with negative or non-integer coefficients.
3. Other structures admitting a dissect scheme: rook polynomials, order invariants, combinatorial geometry, and symmetric functions.
4. In higher dimensions and multi-variables.

References

- J. P. S. Kung, X. Sun, and C. Yan, *Goncarov-Type Polynomials and Applications in Combinatorics*, unpublished manuscript (2006).
- J. P. S. Kung and C. Yan, *Goncarov polynomials and parking functions*, J. Combin. Theory Ser. A **102** (2003), No. 1, 16–37.
- R. Lorentz, S. Tringali and C. Yan. Generalized Goncarov Polynomials, to appear in S. Butler et al. (Eds.), Connections in Discrete Mathematics: A Celebration of the Work of Ron Graham, Cambridge University Press.
- R. Mullin and G.-C. Rota, "On the Foundations of Combinatorial Theory. III. Theory of Binomial Enumeration", in B. Harris (ed.), *Graph Theory and Its Applications*, Academic Press, 1970.
- G.-C. Rota, D. Kahaner, and A. Odlyzko, *On the foundations of combinatorial theory. VII. Finite operator calculus*, J. Math. Anal. Appl. **42** (1973), No. 3, 684–760.