Goncarov Polynomials and Applications in Combinatorics

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Outline

1. Sequences of Biorthogonal Polynomials
   - Goncarov Polynomial and Parking Functions
   - Difference Goncarov Polynomials and Lattice Paths
   - q-Analogues and Combinatorial Enumerators

2. An Application
   - Lattice Paths with Periodic Right Boundary
Biorthogonal polynomials

In $F[x]$,

$$D = \frac{d}{dx}, \quad \varepsilon(a)(f(x)) = f(a).$$

A sequence of linear operators $\varphi_s(D)$ given by

$$\varphi_s(D) = D^s \sum_{r=0}^{\infty} b_{sr} D^r, \quad s = 0, 1, 2, \ldots$$

The polynomial sequences $p_n(x)$ is **biorthogonal to** $\varphi_s(D)$ if

$$\deg(p_n) = n,$$

$$\varepsilon(0) \varphi_s(D)p_n(x) = n! \delta_{sn},$$
Basic properties of $p_n(x)$

- **Determinant formula**

$$p_n(x) = \frac{n!}{b_{00} b_{10} \cdots b_{n0}} \det(M)$$

where

$$M = \begin{bmatrix}
    b_{00} & b_{01} & b_{02} & \ldots & b_{0,n-1} & b_{0n} \\
    0 & b_{10} & b_{11} & \ldots & b_{1,n-2} & b_{1,n-1} \\
    0 & 0 & b_{20} & \ldots & b_{2,n-3} & b_{2,n-2} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \ldots & b_{n-1,0} & b_{n-1,1} \\
    1 & x & \frac{x^2}{2!} & \ldots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!}
\end{bmatrix}.$$
• **Expansion Formula**

\[ p(x) = \sum_{i=0}^{n} \frac{\varepsilon(0) \varphi_i(D) p(x)}{i!} p_i(x), \]

and

\[ x^n = \sum_{i=0}^{n} \frac{n! b_{i,n-i}}{i!} p_i(x). \]

• **Appell relation**

\[ e^{xt} = \sum_{n=0}^{\infty} \frac{p_n(x) \varphi_n(t)}{n!}. \]
Fix a sequence of numbers \((a_0, a_1, a_2, \ldots)\). Let

\[
\varphi_s(D) = e^{a_s D} = \sum_{r=0}^{\infty} \frac{a_s^r}{r!} D^{s+r}.
\]

(Note: in general, \(E^a(f(x)) = f(x + a)\).)

**Goncarov polynomials** \(g_n(x; a_0, a_1, \ldots, a_{n-1})\):

The sequence biorthogonal to \(\varphi_s(D)\)

It is the unique polynomial such that \(\epsilon(a_i)D^i(f(x)) = n!\delta_{n,0}\).
Initial terms

\[ g_0(x) = 1, \]
\[ g_1(x; a_0) = x - a_0 \]
\[ g_2(x; a_0, a_1) = x^2 - 2a_1 x + 2a_0a_1 - a_0^2 \]
\[ g_3(x; a_0, a_1, a_2) = x^3 - 3a_2 x^2 \]
\[ + (6a_1a_2 - 3a_1^2)x - a_0^3 + 3a_0^2a_2 - 6a_0a_1a_2 + 3a_0a_1^2 \]
Basic properties of Goncarov polynomials

1. **Determinant formula** $g_n(x; a_0, \ldots, a_{n-1}) = n! \det(M)$, where

\[
M_{i,j} = \frac{a_i^{j-i}}{(j-i)!} \quad (0 \leq i \leq j \leq n), \quad M_{n,j} = \frac{x^j}{j!}
\]

2. **Linear recurrence**

\[
x^n = \sum_{i=0}^{n} \binom{n}{i} a_i^{n-i} g_i(x; a_0, a_1, \ldots, a_{i-1})
\]

3. **Appell relation**

\[
e^{xt} = \sum_{n=0}^{\infty} \frac{g_n(x; a_0, a_1, \ldots, a_{n-1}) t^n e^{a_n t}}{n!}
\]
Special Properties

- **Differential relation**

\[
D g_n(x; a_0, a_1, \ldots, a_{n-1}) = n g_{n-1}(x; a_1, a_2, \ldots, a_{n-1}),
\]

with initial conditions

\[
g_n(a_0; a_0, a_1, \ldots, a_{n-1}) = \delta_{0n}.
\]

- **Shift invariant formula**

\[
g_n(x + \xi; a_0 + \xi, a_1 + \xi, \ldots, a_{n-1} + \xi) = g_n(x; a_0, a_1, \ldots, a_{n-1}).
\]
Perturbation Formula.

\[ g_n(x; a_0, \ldots, a_{m-1}, a_m + b_m, a_{m+1}, \ldots, a_{n-1}) = g_n(x; a_0, \ldots, a_{m-1}, a_m, a_{m+1}, \ldots, a_{n-1}) - \binom{n}{m} g_{n-m}(a_m + b_m; a_m, a_{m+1}, \ldots, a_{n-1}) g_m(x; a_0, \ldots, a_{m-1}) \]

Binomial expansion

\[ g_n(x + y; a_0, \ldots, a_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(y; a_i, \ldots, a_{n-1}) x^i. \]
Special cases

\[ g_n(x; a, a, \ldots, a) = (x - a)^n \]
\[ g_n(x; y, y + b, y + 2b, \ldots, y + (n - 1)b) = (x - y)(x - y - nb)^{n-1} \]

In particular,

\[ g_n(x; 0, 1, \ldots, n - 1) = x(x - n)^{n-1} \]

The linear recursion gives Abel identity

\[ (x + y)^n = \sum_{i=0}^{n} \binom{n}{i} (y + ib)^{n-i} x(x - ib)^{i-1}. \]
Parking functions

Definition

A sequence of nonnegative integers \((a_1, a_2, \ldots, a_n)\) is a parking function if its nondecreasing rearrangement \(b_1 \leq b_2 \leq \cdots \leq b_n\) satisfy

\[0 \leq b_i \leq i - 1.\]

- \(n = 1\) : (0)
- \(n = 2\) : (0, 0), (0, 1), (1, 0)
- \(n = 3\) : (0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 1), (0, 1, 2)
  and their permutations
Background for Parking Functions

- 1966: (Konheim & Weiss) random hashing functions
- 1970’s: (Foata, Riordan, Knuth) labeled trees and parking functions
- 1990’s: (Stanley) interval orders, hyperplane arrangements, noncrossing partitions, symmetric functions, stochastic processes, and associahedrons, ...
- 2000’s: (Postnikov, Holtz, etc) commutative algebra, polynomial ideals, approximation theory, box spline, ...
General Parking Functions

Let \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) be a sequence of non-decreasing positive integers.

For a sequence \((x_1, x_2, \ldots, x_n)\), the \textit{order statistics} is the nondecreasing rearrangement \( x^{(1)} \leq x^{(2)} \leq \cdots \leq x^{(n)} \).

**Definition**

A \( \mathbf{u} \)-parking function of length \( n \) is a sequence \((x_1, x_2, \ldots, x_n)\) whose order statistics sequence satisfy \( 0 \leq x^{(i)} < u_i \).

When \( \mathbf{u} = (1, 2, \ldots, n) \), we obtain the classical parking functions.
Let $P_n(u)$ be the number of $u$-parking functions of length $n$. For example, for $u = (2, 5)$, $u$-parking functions include

00, 01, 02, 03, 04, 11, 12, 13, 14

and their permutations.

$P_n(u) = 16$
Theorem

\[ P_n(u_1, u_2, \ldots, u_n) = g_n(x; x - u_1, x - u_2, \ldots, x - u_n) = (-1)^n g_n(0; u_1, u_2, \ldots, u_n). \]
Key Fact

It follows from the linear relation

\[ x^n = \sum_{m=0}^{n} (x - u_{m+1})^{n-m} P_m(u_1, u_2, \ldots, u_m). \]

Example: Let \( x = 15 \), and \( u = (1, 3, 5, 7, 9, 11) \).

\[(14, 1, 9, 4, 3, 8) \quad \text{Reorder to} \quad (1, 3, 4, 8, 9, 14) \]
\[\rightarrow \quad (*, 1, *, 4, 3, *) \quad \text{and} \quad (14, *, 9, *, *, 8)\]
Corollaries:

1. $P_n(1, 2, \ldots, n) = (n + 1)^{n-1}$
2. $P_n(a, a + b, a + 2b, \ldots, a + (n - 1)b) = a(a + nb)^{n-1}$
3. $P_n(bu_1, bu_2, \ldots, bu_n) = b^n P_n(u_1, u_2, \ldots, u_n)$.
4. In general, $P_n(u_1, \ldots, u_n) = n! \det(M)$ where $M_{i,j} = u_i^{j-i+1}/(j - i + 1)!$. 
Difference Operator

The backward difference operator

\[ \Delta p(x) = p(x) - p(x - 1), \]

with basis \( x^{(n)} = x(x + 1) \cdots (x + n - 1) \), i.e., \( \Delta x^{(n)} = nx^{(n-1)} \).

Sequence of linear operators \( \psi_s(\Delta) \):

\[ \psi_s(\Delta) = \sum_{r=0}^{\infty} \frac{b_s(b_s + 1) \cdots (b_s + r - 1)}{r!} \Delta^{r+s}, \]

Difference Goncarov polynomial \( \tilde{g}_n(x; b_0, \ldots, b_{n-1}) \):

polynomials biorthogonal to \( \psi_s(\Delta) \)
Properties of Difference Goncarov Polynomials

Again we have

- **Determinant Formula** \( \tilde{g}_n(x; b_0, \ldots, b_{n-1}) = n! \det(N) \), where
  \[
  N_{i,j} = \frac{b_i^{(j-i)}}{(j-i)!} \quad (0 \leq i \leq j \leq n), \\
  N_{n,j} = \frac{x^{(j)}}{j!}
  \]

- **Expansion Formula**
  \[
  p(x) = \sum_{i=0}^{n} \frac{\psi_i(\Delta)(p(x))|_{x=0}}{i!} \tilde{g}_i(x; b_0, \ldots, b_{i-1}).
  \]

- **Linear Recurrence**
  \[
  x^{(n)} = \sum_{i=0}^{n} \binom{n}{i} b_i^{(n-i)} \tilde{g}_i(x; b_0, b_1, \ldots, b_{i-1})
  \]
Appell Relation

\[(1 - t)^{-x} = \sum_{n=0}^{\infty} \tilde{g}_n(x; b_0, b_1, \ldots, b_{n-1}) \frac{t^n}{(1 - t)^{b_n} n!}.\]

Difference relations

\[\Delta \tilde{g}_n(x; b_0, b_1, \ldots, b_{n-1}) = n \tilde{g}_{n-1}(x; b_1, b_2, \ldots, b_{n-1}),\]

with initial conditions

\[\tilde{g}_n(b_0; b_0, b_1, \ldots, b_{n-1}) = \delta_{0n}.\]

Shift invariant formula

\[\tilde{g}_n(x + \xi; b_0 + \xi, b_1 + \xi, \ldots, b_{n-1} + \xi) = g_n(x; b_0, b_1, \ldots, b_{n-1}).\]
Perturbation Formula

Binomial expansion

\[ \tilde{g}_n(x + y; b_0, b_1, \ldots, b_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} \tilde{g}_{n-i}(y; b_i, \ldots, b_{n-1}) x^{(i)}. \]

Special cases.

\[ \tilde{g}_n(x; b, b, \ldots, b) = (x - b)^{(n)} \]

\[ \tilde{g}_n(x; y, y + b, \ldots, y + (n - 1)b) = (x - y)(x - y - nb + 1)^{(n-1)} \]

\[ \tilde{g}_n(0; -1, -2, \ldots, -n) = n!C_n. \]
Lattice path with right boundary

Lattices paths from $(0, 0)$ to $(x - 1, n)$ with steps $(1, 0)$ or $(0, 1) = (x_0, \ldots, x_n)$ where $(x_i, i)$ is the right-most point on the $i$-th row.

**Example**

Lattice path $(2, 5, 7, 10)$.

![Lattice path diagram]
Let $LP_n(b_0, \ldots, b_{n-1})$ be the number of paths from $(0, 0)$ to $(x - 1, n)$ such that $x_i < b_i$ for $0 \leq i \leq n - 1$.

**Theorem**

$$LP_n(b_0, \ldots, b_{n-1}) = \frac{1}{n!} \tilde{g}_n(x; x - b_0, \ldots, x - b_{n-1})$$

$$= \frac{1}{n!} \tilde{g}_n(0; -b_0, \ldots, -b_{n-1}).$$
Sum Enumerator of Parking Functions

An important combinatorial statistic for parking function:

\[ S_n = x_1 + x_2 + \cdots + x_n \quad \text{or} \quad d_n = \binom{n}{2} - S_n. \]

- It is the number of probes in linear hashing
- It is the number of inversions in rooted labeled trees
- It is the distance of an arbitrary region to a basic region in Shi-arrangement
- It is the external activities for labeled trees...

Define the Sum Enumerator

\[ S_n(q; u) = \sum_{(a_1, \ldots, a_n) \in \text{Park}_n(u)} q^{a_1+a_2+\cdots+a_n}. \]
Lemma

\[ [x]^n = \sum_{m=0}^{n} \binom{n}{m} q^{um+1(n-m)} [x - u_{m+1}]^{n-m} S_m(q, u). \]

where \([x] = 1 + q + q^2 + \cdots + q^{x-1} \).

Example: Let \(x = 15\), and \(u = (1, 3, 5, 7, 9, 11)\).

\((14, 1, 9, 4, 3, 8) \rightarrow (\ast, 1, \ast, 4, 3, \ast) \) and \((14, \ast, 9, \ast, \ast, 8)\)
Theorem

\[ S_n(q; u) = P_n([u_1], [u_2], \ldots, [u_n]) = \frac{1}{(1 - q)^n} g_n(1; q^{u_1}, q^{u_2}, \ldots, q^{u_n}) \]

where the q-number \([n] = 1 + q + \cdots + q^{n-1}\).
Some identities of $S_n(q; u)$

- **Linear recursion**

  $$1 = \sum_{m=0}^{n} \binom{n}{m} q^{u_{m+1}(n-m)}(1-q)^m S_m(q; u).$$

- **Appell relation**

  $$e^t = \sum_{n=0}^{\infty} (1-q)^n S_n(q; u) \exp(q^{u_n+1} t) \frac{t^n}{n!}.$$

- **Moments for the sum of general parking functions**
The classical case and Abel Identities

Classical: \( u_i = a + (i - 1)b \), where \( g_n(x; u) = (x - a)(x - a - nb)^{n-1} \).

**Theorem (Knuth)**

*Expected sum of parking functions:*

\[
E_1(n; a, b) = \frac{n(a + 1)}{2} + b \binom{n}{2} - \frac{1}{2} \sum_{i=2}^{n} \binom{n}{i} \frac{i!b^i}{(a + nb)^{i-1}}.
\]
Along the way, we obtain a series of Abel-like identities.

- \[(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} (y + ib)^{n-i} x(x - ib)^{i-1}.\]

- Hurwitz’s version

\[(x + y + nb)^n = \sum_{i=0}^{n} \binom{n}{i} (y + ib)^i x(x + (n - i)b)^{n-i-1}.\]

- \[(a + nb)^{n+1} = \sum_{i=0}^{n} \frac{n!}{(n - i)!} (a + nb)^{n-i} b^i (a + ib).\]

Higher moments can be computed systematically, too!
The reversed sum

Let

\[ d_n(x) = an + b \binom{n}{2} - \sum_{i} x_i. \]

and

\[ D_n(q; a, b) = \sum_{x} q^{d_n(x)} \]

Then

**Theorem**

\[ \sum_{n=0}^{\infty} (q - 1)^n D_n(q; a, b) \frac{t^n}{n!} = \frac{\sum_{n=0}^{\infty} q^{an+b \binom{n}{2}} t^n / n!}{\sum_{n=0}^{\infty} q^{b \binom{n}{2}} t^n / n!}. \]

The case \( a = b = 1 \) is due to Stanley, where \( D_n(q; u) \) is replaced with \( I_n(q) \), the inversion enumerator of labeled trees.
Consider: graphs on $n + 1$ labeled vertices \{0, 1, \ldots, n\}. Let $c(n + 1, k)$ be the number of connected graphs with $n + k$ edges.

**Theorem**

The asymptotic

$$c(n + 1, k) = \rho_{k-1}(n + 1)^{n-1+\frac{3k}{2}}(1 + O(n^{-1/2})).$$

The numbers $\rho_k$ are called Wright Constants.

e.g.,

$$\rho_0 = \frac{\sqrt{2\pi}}{4}, \quad \rho_1 = \frac{5}{24}, \quad \rho_2 = \frac{5\sqrt{2\pi}}{28}.$$
Connected graphs and parking functions

**Key Fact**

\[ c(n + 1, k) = \sum_j p_j \binom{j}{k} \]

where \( p_j \) is the number of ordinary parking functions that have reversed sum \( d_n = j \).
It follows that

**Theorem**

\[ c(n + 1, k) = \frac{(n + 1)^{n-1}}{k!} F_k(n) \]

where \( F_k(n) \) is the expected value of the \( k \)th falling factorial moment of \( d_n \), i.e.,

\[ F_k(n) = \frac{1}{(n + 1)^{n-1}} \sum_x (d_n(x))_k, \]

with \( (x)_k = x(x - 1) \cdots (x - k + 1) \).
Explicit formula for the Wright constants

If $k$ is even and $k = 2l$, then

$$k! \rho_{k-1} = (2l - 1)!!(12)^{-l} + \sum_{s=1}^{l} \binom{2l}{2s} t_{2s} 2^{3s-2} \Gamma(3s - 1)(2l - 2s - 1)!! \frac{(12)^{l-s}}{(12)^{l-s}},$$

and if $K$ is odd and $k = 2l + 1$, then

$$k! \rho_{k-1} = -\sum_{s=0}^{l} \binom{2l + 1}{2s + 1} t_{2s+1} 2^{3s-\frac{1}{2}} \Gamma(3s + \frac{1}{2})(2l - 2s - 1)!! \frac{(12)^{l-s}}{(12)^{l-s}},$$

where $t_r$ satisfy an explicit linear recursion.
Area Enumerator for Lattice Paths

Area enumerator

$$\text{Area}_n(q; b) := \sum_{(x_0, \ldots, x_{n-1}) \in \text{LP}_n(b)} q^{x_0 + x_1 + \cdots + x_{n-1}},$$

Example

Area of the lattice path (2, 5, 7, 10).
Area\(_n(q; b)\) is not a specialization of \(\tilde{g}(x; b_0, \ldots, b_{n-1})\).

\textbf{q-difference Goncarov polynomials:} polynomial sequence biorthogonal to the linear operators

\[
\psi_{q,s}(\Delta_q) = \sum_{s=0}^{\infty} \frac{(b; q)_r}{[r]!} \Delta_q^{r+s},
\]

where \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\) and

\[
\Delta_q p(y) = \frac{p(y) - p(y/q)}{(1 - q)y/q}.
\]
Let $g_n^{(q)}(x; b_0, b_1, \ldots, b_{n-1})$ be the $q$-difference Goncarov polynomials.

**Theorem**

$$\text{Area}_n\left(\frac{1}{q}; b\right) = \frac{(-1)^n}{[n]!(1 - \frac{1}{q})^n} g_n^{(q)}(1; q^{-b_0}, q^{-b_1}, \ldots, q^{-b_{n-1}}).$$

**Remark.** The theory can be extended to parking functions or lattice paths with two-sided boundary.
The right boundary \( s \) is *ultimately periodic* if:
for all but finitely many terms,
\[
  s = s_0, \ldots, s_{k-1}, s_0 + l, \ldots, s_{k-1} + l, s_0 + 2l, \ldots, s_{k-1} + 2l, \ldots
\]

**Example**

Right boundary: \( 2, 5, 2 + 5, 5 + 5, 2 + 10, 5 + 10, \ldots \)
- $s = 1, 2, 3, 4, \ldots$: Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.
- Linear boundary $x = c + dy$

$$LP_n = \frac{c}{c + nd + n} \binom{c + nd + n}{n}.$$ 

- Determinant formula [Steck’69]

$$LP_n(s_0, s_1, \ldots, s_{n-1}) = \det \left[ \begin{array}{cccc} s_i \\ j - i + 1 \end{array} \right]_{0 \leq i, j \leq n-1}.$$ 

- Tennis ball problem [de Mier & Noy, 2004] $s = (E^k N^l)^n$. 

Question

Is \( Q(x) = \sum_n LP_n(s)z^n \) algebraic?
That is, \( Q(x) \) satisfies a polynomial equation, or its coefficients satisfy a recurrence with polynomial coefficients.
Algebraicity

\( y \) is algebraic: \( p_0(x) + p_1(x)y + \cdots + p_d(x)y^d = 0 \)

Fractional power series: \( y = \sum_{n\geq n_0} a_n x^{n/N} \)

\[
K^{\text{fra}}((x)) = K((x))[x^{1/2}, x^{1/3}, x^{1/4}, \ldots]
\]

e.g.

\[
y = x^{1/3} + x^{2/3} + x^{4/3} + x^{5/3} + \cdots,
\]

but not

\[
x^{1/2} + x^{1/3} + x^{1/4} + x^{1/5} + \cdots.
\]

Theorem (Puiseux’s Theorem)

Let \( K \) be an algebraically closed field of characteristic zero. Then the field \( K^{\text{fra}}((x)) \) is an algebraic closure of \( K((x)) \).
Difficulty: How to get an explicit formula or recurrence?
But we already have the Appell relation

\[ \sum_{n=0}^{\infty} \text{LP}_n(s)t^n(1-t)^{s_n} = 1. \]

It is easy for affine boundary \( s_n = c + nd \):

\[ \sum_{n=0}^{\infty} \text{LP}_n(s)z^n = \frac{1}{(1-t)^c}, \]

where \( z = t(1-t)^d \).
How about $s = a, b, a + l, b + l, a + 2l, b + 2l, \ldots$?

Appell relation becomes

$$\sum_{k=0}^{\infty} \text{LP}_2k t^{2k} (1 - t)^{a + kl} + \sum_{k=0}^{\infty} \text{LP}_{2k+1} t^{2k+1} (1 - t)^{b + kl} = 1$$

That is

$$(1 - t)^a Q_0(z) + t (1 - t)^b Q_1(z) = 1,$$

where

$$Q_0(z) = \sum_k \text{LP}_{2k} z^k, \quad Q_1(z) = \sum_{k=0}^{\infty} \text{LP}_{2k+1} z^k, \quad z = t^2 (1 - t)^l.$$
Lemma

Let \( h(t) \) be a power series such that \( h(0) = 1 \). Then the equation

\[
z = t^k h(t)
\]

has \( k \) fractional power series solutions \( \tau_m(z) \), \( 0 \leq m \leq k - 1 \) such that

\[
\tau_0(z) = z^{1/k} + \sum_{i=2}^{\infty} c_i z^{i/k} \quad \text{and} \quad \tau_m(z) = \xi^m z^{1/k} + \sum_{i=2}^{\infty} c_i \xi^{mi} z^{i/k},
\]

for \( 1 \leq m \leq k - 1 \), where \( \xi \) is a primitive \( k \)-th root of unity. Moreover, if \( h(t) \) is algebraic, then \( \tau_0(z), \ldots, \tau_{k-1}(z) \) are also algebraic.
Back to the case $s = a, b, a + l, b + l, a + 2l, b + 2l, \ldots$.

1. By Lemma, $z = t^2(1 - t)^l$ has two algebraic solutions $t_1 = \tau_0(z), t_2 = \tau_1(z)$

2. Appell relation gives

$$
\begin{pmatrix}
(1 - t_1)^a & t_1(1 - t_1)^b \\
(1 - t_2)^a & t_2(1 - t_2)^b
\end{pmatrix}
\begin{pmatrix}
Q_0(z) \\
Q_1(z)
\end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$

3. Solve the above linear system for $Q_0(z)$ and $Q_1(z)$.

4. OGF is $Q_0(z^2) + zQ_1(z^2)$. 
General cases: $s$ is ultimately periodic

1. Appell relation is

$$
\sum_{j=0}^{k-1} Q_j(z) t^j \phi(t)^{b_j} = A(t),
$$

where $Q_j(z) = \sum_{q=0}^\infty \text{LP}_{q+k+j} z^q$.

2. $z = t^k \phi(t)^l$ has $k$ algebraic solutions.

3. Each $Q_j(z)$ is a rational function in the above solutions.

4. OGF is $Q_0(z^k) + zQ_1(z^k) + \cdots + z^{k-1}Q_{k-1}(z^k)$.

Remark: Step $(a, b)$ is also allowed.
Thank you for your attention!

Collaborators: Joseph Kung, Xinyu Sun, Anna de Mier