We study the derangement number on a Ferrers board \( B = (n \times n) - \lambda \) with respect to an initial permutation \( M \), that is, the number of permutations on \( B \) that share no common points with \( M \). We prove that the derangement number is independent of \( M \) if and only if \( \lambda \) is of rectangular shape. We characterize the initial permutations that give the minimal and maximal derangement numbers for a general Ferrers board, and present enumerative results when \( \lambda \) is a rectangle.

**Keywords**: derangement; Ferrers Board

Mathematics Subject Classification: 05A05, 05A18

1. Introduction

The classic derangement question can be formulated on an \( n \times n \) board, where \( n \) is a positive integer. A *permutation* on such an \( n \times n \) board is a placement of points on the board such that no two points are in the same row or column, and every row and column contains one point. This is equivalent to placing \( n \) nonattacking rooks onto the board, as a rook can only move along rows and columns, so if one has two rooks in the same row or column, they must necessarily be attacking one another. Given an initial permutation \( M \), a *derangement* is a permutation on the \( n \times n \) board which shares no common squares with \( M \). It is well-known that the derangement number is independent of the initial permutation \( M \) and only dependent on \( n \). This
is reflected in the enumeration of the derangements
\[ D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}, \]
(1.1)
a classical result that can be dated back to de Montmort [4] and Bernoulli in the early eighteenth century.

This paper investigates the derangement question on a more general class of boards known as Ferrers Boards. Given an \( n \times n \) board as above, let \( r_1, r_2, \ldots, r_n \) denote the rows of the board, where \( r_1 \) is the bottom row. Further, let \( |r_i| \) denote the number of squares in row \( r_i \). Then a Ferrers board \( B \) has the property that \( 1 \leq |r_1| \leq |r_2| \leq \ldots \leq |r_n| \). In this paper we always assume \( |r_n| = n \). Thus, a Ferrers board is an \( n \times n \) board with a section \( \lambda \) missing in the lower right corner, and so we may write \( B = (n \times n) - \lambda \), where \( \lambda \) is an integer partition.

Permutations and derangements on a Ferrers board \( B \) are defined as in the classic case: a permutation is a placement of \( n \) nonattacking rooks on \( B \). Given such a permutation \( M \), a derangement from \( M \) on \( B \) is a permutation on the board that shares no common squares with \( M \).

Denote by \( d_B(M) \) the number of derangements from \( M \) on a board \( B \). Unlike the classical case, the derangement number \( d_B(M) \) for a general Ferrers board \( B \) depends not only on the shape of \( B \), (that is, on \( n \) and \( \lambda \)), but also on the initial permutation \( M \), as shown in Figure 1, where the initial permutation \( M \) and \( M' \) are represented by block dots. In the left figure, there are two derangements, represented by \( R_1 \) and \( R_2 \) respectively. In the right figure, there is only one derangement, represented by \( R \).

Fig. 1: An example showing that \( d_B(M) \) is not constant for all \( M \).

In this paper we will characterize the extremal permutations for which the derangement number reaches a maximum and minimum. We prove that the two extreme cases happen when the permutations correspond to the noncrossing and nonnesting matchings with given left and right endpoints. This is the content of
Section 2. As a corollary, the derangement number is independent of the initial permutation $M$ when the missing section $\lambda$ is of a rectangular shape. The next two sections are devoted to the enumeration of the derangement numbers for the rectangular $\lambda$. In Section 3 we give an explicit formula and express it in terms of an analog of the classical derangement numbers. In Section 4 we establish a recurrence relation, and derive a differential equation satisfied by the generating function of the derangement numbers for the rectangular $\lambda$.

2. Bounds for the Derangement Number on a Ferrers Board

Let $S(B)$ be the set of all permutations on a Ferrers board $B$ (with a given $\lambda$). We label the rows of $B$ ascendingly as $r_1, r_2, \ldots, r_n$ and the columns from left to right as $c_1, \ldots, c_n$. Denote the square in the $i$th row and the $j$th column as $(r_i, c_j)$. Note that $S(B) = \emptyset$ unless the board $B$ contains $\{(r_d, c_d)|1 \leq d \leq n\}$, the diagonal from the lower-left corner to the upper-right corner. We subsequently assume that $B$ does contain that diagonal.

To compare the derangement numbers associated with different initial permutations, we introduce two operations on the permutations on $B$.

2.1. Left-Right and Right-Left Movements

Definition 1 Let $(r_m, c_i), (r_k, c_j) \in B$ where $r_m$ is above $r_k$ (i.e. $m > k$) and $c_i$ is to the left of $c_j$ (i.e. $i < j$). Then the left-right movement is a transformation $LR$ that replaces the squares $(r_m, c_i), (r_k, c_j)$ with $(r_m, c_j), (r_k, c_i)$. That is,

$$LR(\{(r_m, c_i), (r_k, c_j)\}) = \{(r_m, c_j), (r_k, c_i)\}. \quad (2.1)$$

See Figure 2 for an illustration. Note that the rows and columns are not necessarily adjacent.

![Fig. 2: An LR Movement.](image)

We can then define the transformation $RL$, the right-left movement, as the inverse of $LR$, with the additional caveat that if the square $(r_k, c_j)$ is not in $B$, then $RL$ is undefined for the pair $\{(r_m, c_j), (r_k, c_i)\}$.

We will also make the simple observation that given two squares in any permutation $M$, one may be able to perform either a left-right or a right-left movement on the squares, but never both simultaneously.
2.2. Noncrossing and Nonnesting Permutations

Given a Ferrers board $B$, we define two particular permutations using the notation of the previous section: the noncrossing permutation, denoted $NC_B$, and the nonnesting permutation, denoted $NN_B$.

**Definition 2** The noncrossing permutation will be defined algorithmically:

1. Let $(r_1, c_i)$ be the rightmost square of row $r_1$. Then $(r_1, c_i) \in NC_B$.
2. For all subsequent rows $r_m$, let $c_p$ be the rightmost column with $(r_m, c_p) \in B$ that does not contain a square in $NC_B$ yet. Then let $(r_m, c_p) \in NC_B$.

**Definition 3** The nonnesting permutation is defined as $NN_B = \{(r_i, c_i) | 1 \leq i \leq n\}$. This corresponds to the diagonal from the lower left to the upper right on $B$.

Referring to the example in Figure 1, the permutation $M$ in the left figure is $NN_B$, and the one in the right figure is $NC_B$.

These definitions come from the well-known noncrossing and nonnesting matchings. Permutations on the set of Ferrers boards of the shapes $\{(n \times n) - \lambda : \lambda \text{ an integer partition}\}$ are in one-to-one correspondence with the set of complete matchings of $\{1, 2, \ldots, 2n\}$, while permutations on a fixed board $B$ correspond to matchings with given sets of minimal elements and maximal elements, as described in [3] and [10, Sec4.2].

To see the connection, given the board $B$, starting from the lower-left corner, travel along its southeast boundary until the upper-right corner is reached, and label the steps from 1 to $2n$. Let $S$ be the set of labels on the horizontal steps, and $T$ be the set of labels on the vertical steps. Then a permutation on $B$ can be represented as a matching between $S$ and $T$, where $(r_i, c_j)$ is in the permutation if and only if the label of row $r_i$ is matched with the label of column $c_j$. Then $NC_B$ and $NN_B$ correspond exactly to the unique noncrossing and nonnesting matchings from $S$ to $T$. See Figures 3 and 4 for illustrations.

![Fig. 3: Equivalence between Noncrossing Permutation and Noncrossing Matching](image-url)
Noncrossing and nonnesting matchings are special cases of noncrossing/nonnesting set partitions, which are among the most fundamental structures in combinatorics, and play important roles in other disciplines, for example, in free probability theory [7], representation of Coxeter groups [1], and RNA molecule structures [8], to list a few.

2.3. Connecting LR, RL, NC\(_B\) and NN\(_B\)

We establish two lemmas that connect the previous notions together.

**Lemma 2.1.** The noncrossing permutation is a permutation such that initially only left-right movements can be performed on its elements. Similarly, the nonnesting permutation is a permutation such that initially only right-left movements can be performed on its elements.

**Proof.** We begin with NC\(_B\). Assume \((r_m, c_i), (r_k, c_j) \in NC_B\) with \(m > k\). If \(i < j\), then clearly an LR movement can be performed. On the other hand, if \(i > j\), we show that the square \((r_k, c_i) \notin B\), hence it is not possible to perform a RL movement. Prove by contradiction. Assume \((r_k, c_i) \in B\). Then, since \((r_k, c_j) \in NC_B\), we have that \((r_k, c_l)\) cannot be the rightmost square in row \(r_k\) not already in NC\(_B\) for \(j < t \leq i\). But we have that every such \((r_k, c_l) \in B\), hence for every column \(c_l\), there must be a row \(r_h\) (with \(h < m\)) such that \((r_h, c_l) \in NC_B\). In particular, there is a row \(r_l\), such that \((r_l, c_i) \in NC_B\), and \(l < k < m\), contradiction to the assumption that \((r_m, c_i) \in NC_B\).

For the NN\(_B\) permutation, let \((r_m, c_m), (r_n, c_n) \in NN_B\), with \(m < n\). Then, by comparison to Definition 1, the conditions for a LR movement are nowhere satisfied, but the conditions for a RL movement are satisfied if \((r_m, c_i) \in B\). Thus, only RL movements can be initially performed on the NN\(_B\) permutation.

**Lemma 2.2.** Let \(M\) be any permutation on \(B\). Then, \(M\) can be attained by a
sequence of LR movements starting from \( NC_B \). Similarly, \( M \) can also be attained by a sequence of RL movements starting from \( NN_B \).

**Proof.** We present an algorithm for \( NC_B \) to attain any other permutation \( M = \{(r_m, c_{i_m})|1 \leq m \leq n\} \) by a sequence of LR movements. This algorithm will operate from \( m = 1 \) to \( m = n \). For \( r_1 \):

1. Let \((r_1, c_{j_1}) \in NC_B \). By definition, it is the rightmost cell of \( r_1 \). If \( c_{i_1} = c_{j_1} \), then move onto the row \( r_2 \) as described below. Otherwise, go to step 2.
2. Let \( C_{1j} \) be the set of columns between \( c_{i_1} \) and \( c_{j_1} \). Take the maximal element less than \( j_1 \), say \( k \), of \( C_{1j} \). Assume that \((r_{ik}, c_k) \in NC_B \) for some row \( r_{ik} \). Then, do:

\[
LR(\{(r_1, c_{i_1}), (r_{ik}, c_k)\}) = \{(r_1, c_k), (r_{ik}, c_{i_1})\}.
\]

Update \( C_{1j} \) by deleting \( k \) from it.
3. With the new element \((r_1, c_k)\), continue to find a maximal element \( t \) of \( C_{1j} \) and a square \((r_{it}, c_t) \in NC_B \). As in step 2, perform the LR movement on the two squares \((r_1, c_k), (r_{it}, c_t)\). Continue to do this until the square \((r_1, c_{i_1})\) is reached. Then proceed to \( r_2 \) as described below.

For the rows \( r_m \), \( m \geq 2 \):

1. In general, apply the algorithm as above. However, when forming the set \( C_{mj} \) of columns between \( c_{i_m} \) and \( c_{m_m} \), where \((r_m, c_{jm})\) is a square attained at some time during the performance of the algorithm on the \( m \)th row, ignore columns in the set \( \{c_{ia} : a < m, (r_a, c_{ia}) \in M\} \).
2. Apply LR movement to \((r_m, c_{jm})\) and \((r_{it}, c_t)\) where \( c_t \) is the rightmost column in \( C_{mj} \).
3. Continue this procedure until a permutation with the squares \((r_1, c_{i_1}), \ldots, (r_m, c_{im})\) is attained. Then do the same procedure for the row \( r_{m+1} \).

After a square \((r_m, c_{im})\) is attained in the above algorithm, it is subsequently fixed during the succeeding operations. A crucial observation is that after the procedure on row \( r_1 \), the squares of \( NC_B - \{(r_1, c_{i_1})\} \) are moved to \( NC_B_1 \), where \( B_1 \) is the board obtained from \( B \) by removing the first row \( r_1 \) and the column \( c_{i_1} \). Hence by induction any permutation \( M \) can be generated from a sequence of LR movements starting from \( NC_B \).

The algorithm for \( NN_B \) is similar. The algorithm computes a minimal leftmost column greater than the current column with a row greater than the current row at the given stage of the process. Then, an RL movement is performed on the two generated points, and the same process is applied onto the newly generated permutation until the desired \((r_m, c_{im})\) is achieved for a given row \( r_m \). As in the algorithm for \( NC_B \), those points remain fixed. Hence, any permutation \( M \) can be achieved by a sequence of RL movements from \( NN_B \).
2.4. Producing a Bound for the Derangement Number

Let \( M \in S(B) \). We now present and prove a general bound for \( d_B(M) \).

**Theorem 2.4.** If \( M \in S(B) \), and \( NC_B \) and \( NN_B \) are the noncrossing and nonnesting permutations on \( B \), then the following inequality holds:

\[
d_B(NC_B) \leq d_B(M) \leq d_B(NN_B).
\]

First, we will establish a lemma.

**Lemma 2.5.** Let \( (r_m, c_i), (r_k, c_j) \in B \) with \( m > k \) and \( i < j \). Let \( M, M' \in S(B) \) so that \( M \) contains \( (r_m, c_i) \) and \( (r_k, c_j) \), and \( M' \) be obtained from \( M \) by \( LR(\{(r_m, c_i), (r_k, c_j)\}) \). Let \( D_B(M) \) denote the set of all derangements on \( B \) from the permutation \( M \), and let \( D_B(M') \) denote the set of all derangements on \( B \) from the permutation \( M' \). Then there is an injection from \( D_B(M) \) to \( D_B(M') \). It follows that \( d_B(M) \leq d_B(M') \).

**Proof.** We will define an injective map \( \phi : D_B(M) \rightarrow D_B(M') \), and demonstrate that in certain cases the map fails to be surjective.

We partition the derangements of \( D_B(M) \) into three disjoint sets: let \( S_i(M) \) be the set of all derangements from \( M \) on \( B \) that contains exactly \( i \) squares from \( \{(r_k, c_j), (r_m, c_i)\} \), for \( i = 0, 1, 2 \). Further divide \( S_i(M) \) into two subsets: \( S_{ia} \), the subset of all derangements which contain \( (r_m, c_i) \), but not \( (r_k, c_j) \); and \( S_{ib} \), the subset of all derangements which contain \( (r_k, c_j) \), but not \( (r_m, c_i) \).

We define a map \( \phi : D_B(M) \rightarrow D_B(M') \) as follows:

- For \( \rho \in S_0(M) \), set \( \phi(\rho) = \rho \).
- For \( \rho_2 \in S_2(M) \), set
  \[
  \phi(\rho_2) = \rho_2 \cup \{(r_m, c_i), (r_k, c_j)\} - \{(r_m, c_i), (r_k, c_j)\}.
  \]
- For \( \rho_1 \in S_{ia} \), let \( (r_k, c_j) \) be the square of \( \rho_1 \) in the row \( r_k \). Set
  \[
  \phi(\rho_1) = \rho_1 \cup \{(r_m, c_i), (r_k, c_j)\} - \{(r_m, c_i), (r_k, c_j)\}.
  \]

The square \( (r_m, c_i) \in B \) because \( B \) is a Ferrers board and \( m > k \).

For \( \rho'_1 \in S_{ib} \), we have that \( (r_k, c_j) \in \rho'_1 \) but \( (r_m, c_i) \notin \rho'_1 \). Let \( (r_m, c_i) \) and \( (r_d, c_j) \) be the squares of \( \rho'_1 \) that lie in row \( r_m \) and column \( c_j \) respectively. Then set

\[
\phi(\rho'_1) = \begin{cases} 
\rho'_1 \cup \{(r_k, c_j), (r_d, c_j)\} - \{(r_k, c_j), (r_d, c_j)\} & \text{if } (r_k, c_j) \notin B, \\
\rho'_1 \cup \{(r_m, c_i), (r_k, c_j), (r_d, c_j)\} - \{(r_m, c_i), (r_k, c_j), (r_d, c_j)\} & \text{if } (r_k, c_j) \in B.
\end{cases}
\]
See Figure 5 for a possible arrangement of $r_m, r_k, r_d$ and $c_i, c_j, c_e$.

We show that $\phi$ defined in this way is injective. Note that $\phi$ maps $S_i(M)$ to $S_i(M')$, and is invertible when restricted to $S_0(M)$ and $S_2(M)$. Hence we only need to check $\phi$ on $S_1(M)$.

By definition, $\phi$ maps $S_1(M)$ to derangements $\alpha$ of $M'$ such that $\alpha$ contains $(r_k, c_j)$ but not $(r_m, c_i)$, and the square $(r_k, c_e)$ is in $B$, if $(r_m, c_e)$ is the square of $\alpha$ in row $r_m$. On the other hand, $\phi$ maps $S_1(M)$ to derangements $\beta$ of $M'$ such that

- either $\beta$ contains $(r_m, c_i)$ but not $(r_k, c_j)$,
- or $\beta$ contains $(r_k, c_j)$, but not $(r_m, c_i)$, and the square $(r_k, c_e)$ is not in $B$, where $(r_m, c_e)$ is the square of $\beta$ in row $r_m$.

Hence no two derangements of $D_B(M)$ have the same image and $\phi$ is injective.

To see that the map $\phi$ is not necessarily surjective, consider a board with rows $r_d, r_k, r_m$ ($d < k < m$) and columns $c_i, c_j, c_e$ ($i < j < e$), where $(r_m, c_e), (r_k, c_j), (r_d, c_i) \in B$ but $(r_d, c_j), (r_d, c_e), (r_k, c_e) \notin B$. (See Figure 5, where block dots are squares of $M'$ and $R$'s belong to a derangement of $M'$.) Then any derangement containing $(r_m, c_e), (r_k, c_j), (r_d, c_i)$ is not an image of $\phi$. In this case $d_B(M) < d_B(M')$.

\[\begin{array}{ccc}
     & c_i & c_j & c_e \\
     r_m & \bullet & R & \\
     r_k & R & \\
     r_d & R & \lambda \\
\end{array}\]

Fig. 5: The map $\phi$ may not be surjective.

**Proof of Theorem 2.4.** By Lemma 2.2, any permutation $M$ can be generated from a sequence of left-right moves beginning with $NC_B$, hence that $d_B(NC_B) \leq d_B(M)$. Again from Lemma 2.2 any permutation $M$ can be achieved from a sequence of right-left moves starting from $NN_B$, hence that $d_B(M) \leq d_B(NN_B)$.

There is now a corollary which will aid the enumeration of the derangement numbers.

**Corollary 2.6.** If $\lambda$ is rectangular, then $d_B(M)$ is independent of the initial choice of $M$ (corresponding to a property in the classic derangement case). The following
converse also holds: if $n$ is sufficiently large, and $d_B(M)$ is independent of $M$, then $\lambda$ must be rectangular.

**Proof.** The proof of Lemma 2.5 demonstrates that the function $\phi$ is bijective unless each of $(r_d, c_j), (r_d, c_e), (r_k, c_e) \in \lambda$, which is impossible if $\lambda$ is rectangular. Thus if $\lambda$ is rectangular, $\phi$ is bijective and so $d_B(M)$ is independent of the choice of $M$.

To prove the converse, suppose that $\lambda$ is nonrectangular. From the proof of Lemma 2.5 and Figure 5, it is sufficient to show that when $n$ is sufficiently large, there exist two permutations $M'$ and $\rho$ on $B$, and rows $r_d, r_k, r_m$, columns $c_1, c_j, c_e$ such that

1. $d < k < m$ and $i < j < e$,
2. $(r_m, c_j), (r_k, c_j), (r_d, c_i) \in B$ but $(r_d, c_e), (r_k, c_e) \notin B$.
3. $(r_m, c_j), (r_k, c_j) \in M'$, and $(r_m, c_e), (r_k, c_j), (r_d, c_i) \in \rho$.

Assume the minimal rectangle containing $\lambda$ is of size $a \times b$. Since $\lambda$ is nonrectangular, we have $a \geq 2$ and $b \geq 2$. Let $n \geq a + b$. We will construct the desired permutations $M'$ and $\rho$ in $B = (n \times n) - \lambda$ by the following steps.

**Step 1.** Take a derangement $\rho_1$ in the first $a$ rows and $a$ columns containing the square $R_1 = (r_1, c_a)$. Take a derangement $\rho_2$ in the last $(n - a)$ rows and $n - a$ columns containing the square $R_2 = (r_{a+1}, c_n)$.

**Step 2.** Assume $R_3 = (r_f, c_t)$ is a corner of the board $B$ such that $1 < f \leq a$ and $|r_{f-1}| < |r_f| < |r_{a+1}| = n$. Let $(r_f, c_{f_1})$ be the square of $\rho_1$ in row $r_f$ and $(r_{t_1}, c_t)$ be the square of $\rho_2$ in column $c_t$. Set

$$\rho = \rho_1 \cup \rho_2 \cup \{(r_f, c_t), (r_{t_1}, c_{f_1})\} - \{(r_f, c_{f_1}), (r_{t_1}, c_t)\}.$$ 

Then $\rho$ is a permutation on the board $B$ containing squares $R_1, R_2, R_3$.

**Step 3.** In the lower-left $a \times a$ sub-board of $B$ there are only $a - 1$ squares belonging to $\rho$. Take a permutation $M_1$ of size $a$ that contains $(r_f, c_a)$ and avoids squares of $\rho$. Similarly, in the upper-right $(n - a) \times (n - a)$ sub-board of $B$ there are $n - a - 1$ square belonging to $\rho$. Take a permutation $M_2$ of size $n - a$ that contains $(r_{a+1}, c_t)$ and avoids squares of $\rho$. Finally set $M' = M_1 \cup M_2$.

See Example 1 for an illustration with $a = b = 3$ and $n = 6$. The permutations $M'$ and $\rho$ satisfy the desired conditions with the involved rows $r_1, r_f, r_{a+1}$ and columns $c_a, c_t, c_n$. The existence of permutations $\rho_1, \rho_2, M_1, M_2$ in Steps 1 and 3 follows from the following property of classical derangements: Given a permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$ of length $n \geq 2$, then there are $D_n/(n-1) = D_{n-1} + D_{n-2}$ derangements $\sigma$ from $\pi$ with $\sigma_k = k$, for all $k \not= \pi_1$. See, for example, [2, Chap 6.3]

**Example 1.** Fig. 6 shows the construction of $M'$ and $\rho$ for a board, where $a = b = 3$ and $n = 6$. In the first figure, $\rho_1$ and $\rho_2$ are represented by $R$ where $R_1 = (1, 3)$ and $R_2 = (4, 6)$. The second figure shows the permutation $\rho$, where the two new squares contain boldface $Rs$. In the third figure the black dots represent $M'$. The involved rows and columns are $r_1, r_2, r_4$ and $c_3, c_4, c_6$. 


3. Enumeration of Derangements in Ferrers Board with a Missing Rectangular Section

The exact enumeration of the derangement numbers may be described as a rook placement on a board with given forbidden positions. For a Ferrers board as we have defined it, the forbidden positions are the squares in $F = \lambda \cup M$, where $\lambda$ is the missing section and $M$ is the initial permutation. There is a well-known theorem connecting the rook coefficients $r_k$ and the number of permutations avoiding given forbidden positions. The theorem can be found in [9]; the result is due to Kaplansky and Riordan in [5].

**Theorem 3.1.** Let $N_0$ be the number of ways to place $n$ nonattacking rooks on an $n \times n$ board avoiding the forbidden positions $F$. Then, the following equality holds:

$$N_0 = \sum_{k=0}^{n} r_k (-1)^k (n-k)!$$

where the rook coefficient $r_k$ is the number of ways to place $k$ nonattacking rooks onto the forbidden positions $F$.

This theorem follows from the Principle of Inclusion-Exclusion. We will investigate specifically the cases where $\lambda$ is rectangular. In the rest of the paper we will assume that $\lambda = r \times s$ for some fixed $r, s \geq 0$.

### 3.1. Rook Coefficients for a Rectangular $\lambda$

Let $F = \lambda \cup M$ be the set of forbidden positions. We will first compute the rook coefficients for $F$. 
Proposition 3.2. Let $r_k$ be the $k$th rook coefficient for the forbidden area $F$. Then,

$$ r_k = \sum_{i=0}^{k} \binom{r}{i} \binom{s}{i} i! \binom{n - 2i}{k - i} $$

Proof. The two sections of the forbidden positions, $\lambda$ and $M$, are disjoint, and so we may place points on $\lambda$ first, and then points on $M$. Suppose that we place $i$ points on $\lambda$ and $k - i$ points on $M$. The number of ways to place $i$ points on $\lambda$ is denoted $r_{i,\lambda}$, analogously to the other rook coefficients. Then, note that each of the $i$ points on $\lambda$ “attacks” two points of $M$ (they attack a particular row and a particular column). Since no two of the $i$ points on $\lambda$ can be in the same row or column, there are then $2i$ points on $M$ that are attacked. The number of ways to choose $k - i$ points on $M$ from $n - 2i$ possible points is given by the binomial coefficient $\binom{n - 2i}{k - i}$. Hence, summing over all possible $i$, we have that

$$ r_k = \sum_{i=0}^{k} r_{i,\lambda} \binom{n - 2i}{k - i} $$

We now need to merely compute $r_{i,\lambda}$. But this is easily done. Since $\lambda = r \times s$ is rectangular, choose $i$ of the $r$ rows and $i$ of the $s$ columns on which to place points. If the rows are fixed, then there are $i!$ possible permutation of the columns with those rows. Hence, $r_{i,\lambda} = \binom{r}{i} \binom{s}{i} i!$, and so

$$ r_k = \sum_{i=0}^{k} \binom{r}{i} \binom{s}{i} i! \binom{n - 2i}{k - i} $$

as desired. \qed

Using an Abel-type inverse relation, we derive from Proposition 3.2 a recurrence relation for $r_k$.

Proposition 3.3. The rook coefficients $r_k$ for the forbidden area $F$ satisfy the recurrence

$$ \binom{r}{k} \binom{s}{k} k! = \sum_{i=0}^{n} \frac{n - 2i}{n - 2k} \binom{-n + 2k}{k - i} r_i. $$

Proof. Let $t_i = \binom{r}{i} \binom{s}{i} i!$. From Proposition 3.2 we have $r_k = \sum_{i=0}^{k} A_{k,i} t_i$ where $A_{k,i} = \binom{n - 2i}{k - i}$. Hence $t_k = \sum_{i=0}^{k} B_{k,i} r_i$ where $(B_{k,i})$ is the inverse matrix of $(A_{k,i})$.

The formula of $B_{k,i}$ can be found in [6, Ex 3.1(28)], which states that if $A_{k,i} = \binom{c + kp + iq}{c + kq}$, then the entries of the inverse matrix are given by

$$ B_{k,i} = \frac{[c + k(p + q)][c + i(p + q)] + (k - i) pq + (c + kp + iq)}{(c + ip + kq)(c + ip + kq + 1)} \binom{-c - ip - kq}{k - i}. $$
Substituting \( c = n, p = 0 \) and \( q = -2 \), we obtain
\[
B_{k,i} = \frac{n - 2i}{n - 2k} \left( -n + 2k \right),
\]
which proves the recurrence.

We will now compute the generating function for the rook coefficients
\[
\sum_{k=0}^{n} r_k x^k.
\]

**Theorem 3.4.** Let \( R(x) = \sum_{k=0}^{n} r_k x^k \) be the ordinary generating function for the rook coefficients as above. Then,
\[
R(x) = \sum_{k=0}^{n} r_k x^k = (1 + x)^n \sum_{i=0}^{\min(r,s)} \binom{r}{i} \binom{s}{i} i! \left( \frac{x}{1 + x} \right)^i,
\]
for sufficiently large \( n \).

**Proof.** Using the formula for the rook coefficients and some well-known generating functions, we obtain
\[
R(x) = \sum_{k=0}^{n} r_k x^k = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{r}{i} \binom{s}{i} i! \binom{n - 2i}{k - i} x^k
\]
\[
= \sum_{i=0}^{n} \binom{r}{i} \binom{s}{i} i! x^i \sum_{k=i}^{n} \binom{n - 2i}{k - i} x^{k-i}
\]
\[
= \sum_{i=0}^{n} \binom{r}{i} \binom{s}{i} i! x^i (1 + x)^{n-2i}
\]
\[
= (1 + x)^n \sum_{i=0}^{\min(r,s)} \binom{r}{i} \binom{s}{i} i! \left( \frac{x}{1 + x} \right)^i.
\]

For \( i > \min(r,s) \), note that \( \binom{r}{i} \binom{s}{i} = 0 \), so the sum is 0 for such \( i \). Hence, the desired equality is attained.

**3.2. An Explicit Formula for the Derangement Numbers for Rectangular \( \lambda \)**

Let \( d_{n,r,s} = d_B(M) \) denote the number of derangements on the Ferrers board \( B = (n \times n) - (r \times s) \), which is independent of \( M \). Applying Theorem 3.1 we get
\[
d_{n,r,s} = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{r}{i} \binom{s}{i} i! \binom{n - 2i}{k - i} (-1)^k (n-k)!.
\]
We will present an equivalent form of this sum in which $d_{n,r,s}$ is expressed as a finite sum of $\min(r,s) + 1$ terms. To do so, we need to introduce a new kind of derangement number, which is a straightforward generalization of the classical derangement number $D_n$.

For $q \leq p$, let $D_{p,q}$ denote the number of permutations of length $p$ such that the first $q$ numbers satisfy the derangement property: i.e. 1 is not in the first position, 2 is not in the second position, ..., $q$ is not in the $q$th position. Clearly when $p = q$ we get the classical derangement numbers. In general, we have the following formula.

**Proposition 3.5.**  
$$D_{p,q} = \sum_{i=0}^{q} (-1)^i \binom{q}{i} (p-i)!$$

**Proof.** This is a direct computation from the principle of inclusion-exclusion, and closely resembles the computation for the classic derangement numbers. Let $P_i$ be the property that $a_i = i$ (where each permutation is written $a_1a_2 \ldots a_p$). Suppose that $j$ of the $P_i$ are satisfied. Then, there are $\binom{q}{j}$ ways to select $j$ properties, and $(p-j)!$ ways to permute the other elements. Hence, by PIE, summing over all possible $j$, we obtain

$$D_{p,q} = \sum_{j=0}^{q} (-1)^j \binom{q}{j} (p-j)!$$

as desired. \hfill \square

We will now establish an equivalent form of $d_{n,r,s}$ in terms of $D_{p,q}$.

**Proposition 3.6.**  
$$d_{n,r,s} = \sum_{i=0}^{\min(r,s)} (-1)^i i! \binom{r}{i} \binom{s}{i} D_{n-i,n-2i} \text{ for sufficiently large } n.$$  

**Proof.** We compare to the form given above and show the two are equivalent. To wit:

$$d_{n,r,s} = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{r}{i} \binom{s}{i} i! \binom{n-2i}{k-i} (-1)^k (n-k)!$$

$$= \sum_{i=0}^{n} \binom{r}{i} \binom{s}{i} i! \sum_{k=i}^{n} \binom{n-2i}{k-i} (-1)^k (n-k)!$$

$$= \sum_{i=0}^{n} \binom{r}{i} \binom{s}{i} i! \sum_{k=i}^{n-i} \binom{n-2i}{k} (-1)^{k+i} (n-(k+i)!)$$

$$= \sum_{i=0}^{n} \binom{r}{i} \binom{s}{i} i! (-1)^i D_{n-i,n-2i}.$$
As before, if \( i > \min(r, s) \), then \((\cdot)_i^i\) = 0, so the sum can be displayed in the desired form if \( n \geq \min(r, s) \).

This form is interesting because the numbers \( D_{p,q} \) have an easily computable exponential generating function, similar to the classical derangement numbers.

### 3.3. A Recurrence for the Derangement Numbers

In this subsection, we present a linear recurrence and partial differential equation satisfied by the derangements numbers \( d_{n,r,s} \) associated with the Ferrers board \( B = (n \times n) - (r \times s) \).

**Theorem 3.7.** The derangement numbers \( d_{n,r,s} \) satisfy the following linear recurrence:

\[
d_{n,r,s} = (n-r-s-1)(d_{n-1,r,s} + d_{n-2,r,s}) + s(d_{n-1,r,s} + d_{n-2,r,s-1}) + r(d_{n-1,r-1,s} + d_{n-2,r-1,s})
\]  

for \( n > r + s \), and \( d_{r+s+r,s} = D_rD_s \), where \( D_n \) is the classical derangement number given in (1.1). By convention, we have \( d_{n,r,s} = 0 \) if \( n \leq 1 \) or \( r, s < 0 \) or \( n < r + s \).

**Proof.** As \( \lambda \) is rectangular, by Corollary 2.6 the number of derangements from any initial permutation on \( B \) is constant. Therefore, we are free to choose any permutation \( M \) on \( B \) from which to compute \( d_B(M) \), so we choose \( M = NC_B \). For \( n \geq r + s \), in reference to Figure 7, we see that the noncrossing permutations \( NC_B \) has the following pattern: the first \( r \) squares proceed diagonally to the northwest from \((r_1, c_{n-s})\); directly above the \( r \)-th row are \( s \) squares proceeding diagonally to the northwest from \((r_{r+1}, c_n)\), and the remaining squares are again diagonal and to the left of the first \( r \) squares of \( NC_B \).

When \( n = r + s \), a derangement in \( D_B(NC_B) \) is a union of a derangement in the lower-left \( r \times r \) sub-board avoiding the southeast-northwest diagonal, together with a derangement in the upper-right \( s \times s \) sub-board avoiding the southeast-northwest diagonal. Hence \( d_{r+s+r,s} = D_rD_s \).

Now assume \( n > r + s \). Let \( A \) be the square of a derangement \( \rho \) in \( D_B(NC_B) \) in the first column. To establish the recurrence, we discuss by cases based on the location of \( A \). Assume \( A \) is in row \( r_a \). These are the three cases.

The first case is that \( A \) is in one of the top \( n-r-s \) rows, i.e., \( a > r + s \). We consider the square of \( \rho \) in the last row \( r_n \). There are two possibilities: \((r_n, c_n) \not\in \rho \) or \((r_n, c_a) \in \rho \). If \((r_n, c_n) \not\in \rho \), removing column \( c_1 \) and row \( r_a \), we have a board \( B_1 = (n-1) \times (n-1) - r \times s \). The derangement \( \rho \), when restricted to \( B_1 \), is a derangement avoiding \( (NC_B \cap B_1) \cup \{(r_n, c_n)\} \). There are \( d_{n-1,r,s} \) such derangements, On the other hand, if \((r_n, c_a) \in \rho \), we can remove rows \( r_a, r_n \) and columns \( c_1, c_a \) to obtain a board \( B_2 \) of size \((n-2) \times (n-2) - r \times s \), with a permutation \( NC_B \cap B_2 \) forbidden.
By hypothesis there are \( d_{n-2,r,s} \) derangements on \( B_2 \). Combining the two cases and noting that there are \( n - r - s - 1 \) choices for \( r_a \) as \( a \neq n \), we have that there are \((n - r - s - 1)(d_{n-1,r,s} + d_{n-2,r,s})\) derangements when \( a > r + s \).

The next two cases are that \( A \) is in one of the middle \( s \) rows, or \( A \) is in one of the bottom \( r \) rows. That is, \( r < a \leq r + s \) or \( a \leq r \). A similar discussion on whether \( \rho \) contains the square \((r_n,c_a)\) leads to the terms \( s(d_{n-2,r,s-1} + d_{n-1,r,s}) \) for the second case, and the terms \( r(d_{n-2,r-1,s} + d_{n-1,r-1,s}) \) for the third case.

Combining these three cases gives the desired recurrence.

**Remark.** It is clear that \( d_{n,r,s} = D_n \) if \( r \) or \( s \) is zero. In that case the recurrence (3.1) reduces to the well-known equation

\[
D_n = (n - 1)(D_{n-1} + D_{n-2}).
\]

Although a linear recurrence has been established, there does not seem to be a simple formula for a multivariate generating function of \( d_{n,r,s} \). Nevertheless, let

\[
D(x,y,z) = \sum_{r,s \geq 0} \sum_{n \geq r+s} d_{n,r,s} x^n \frac{y^r z^s}{r! s!}.
\]

There is a partial differential equation which such a generating function would have to satisfy.

**Proposition 3.8.** Let \( D(x,y,z) \) be defined as above. Then, \( D(x,y,z) \) satisfies the
following partial differential equation:

$$(1 - xy)T(x, y, z) + (x^2 y + x^2 z + xy + x^2 - 1)D(x, y, z)$$

$$+ (x^3 + x^2) \frac{\partial D}{\partial x} = (xy + x^2 y) \frac{\partial D}{\partial y} - x^2 z \frac{\partial D}{\partial z} = 0,$$

where

$$T(x, y, z) = \frac{e^{-xy-xz}}{(1 - xy)(1 - xz)}.$$

**Proof.** Sketch of Proof. First note that when $n = r + s$,

$$\sum_{r, s \geq 0} d_{x+r,y+s} x^r y^s \frac{z^s}{r! s!} = \sum_{r, s \geq 0} D_r D_s (xy)^r (xz)^s \frac{e^{-xy-xz}}{r! s!} = T(x, y, z),$$

where we use the classical result

$$\sum_{n \geq 0} \frac{D(n)x^n}{n!} = \frac{e^{-x}}{1-x}.$$

Hence

$$D(x, y, z) = T(x, y, z) + \sum_{r, s \geq 0} \sum_{n>r+s} d_{n,r,s} x^n y^r z^s \frac{1}{r! s!}.$$

Applying the recurrence (3.1) to $d_{n,r,s}$ in the second summation, and using that

$$\frac{\partial D}{\partial x} = \sum_{r, s \geq 0} \sum_{n \geq r+s} n d_{n,r,s} x^{n-1} y^r z^s \frac{r^s}{r! s!} = \sum_{r, s \geq 0} \sum_{n > r+s} (n-1) d_{n-1,r,s} x^{n-2} y^r z^s \frac{1}{r! s!}$$

$$\frac{\partial D}{\partial y} = \sum_{r \geq 1, s \geq 0} \sum_{n \geq r+s} d_{n,r,s} x^n y^{r-1} z^s \frac{1}{(r-1)! s!}$$

$$\frac{\partial D}{\partial z} = \sum_{r > 0, s \geq 1} \sum_{n \geq r+s} d_{n,r,s} x^n y^r z^{s-1} \frac{1}{r! (s-1)!},$$

together with

$$\sum_{r, s \geq 0} \sum_{n > r+s} r d_{n-1,r-1,s} x^n y^r z^s \frac{1}{r! s!}$$

$$= \sum_{r, s \geq 0} \sum_{n > r+s} d_{n,r,s} x^{n+1} y^r z^s \frac{1}{r! s!}$$

$$= \sum_{r, s \geq 0} \sum_{n > r+s} d_{n,r,s} x^{n+1} y^r z^s \frac{1}{r! s!} - \sum_{r, s \geq 0} D_r D_s x^{r+s+1} y^r z^s \frac{1}{r! s!}$$

$$= xyD(x, y, z) - xyT(x, y, z),$$

we obtain the desired equation. □
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