Parking Distributions on Trees

Catherine Yan
joint work with
Steve Butler, Ron Graham

Department of Mathematics
Texas A&M University

AMS Fall Central Sectional Meeting, September 2017

Classical Parking Functions

Classical parking process :

i) n parking spaces labeled $1, 2, \ldots, n$ along a street.

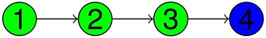
ii) n cars enter the street one by one, each with a preferred space.

iii) Each car goes directly to its preferred space and parks there if the space is empty, otherwise moves towards the exit and takes the first available space.

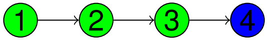
iv) If there is no space available, the car exits.

Definition

A parking function of length n is a preference sequence for the cars in which all cars are able to park.



Parking functions: 1241, 3112, 1233, etc.
Not a parking function: 2224



Parking functions: 1241, 3112, 1233, etc.
Not a parking function: 2224

Basic Counting

PFs of length $n = (n+1)^n$,
increasing PFs of length $n = \frac{1}{n+1}\binom{2n}{n}$.

Three Essential Features

- Each car has an initial preferred parking spot and goes directly to this spot.
- If a spot is currently occupied then cars have a consistent rule to proceed and search for an opening.
- Each car will terminate its search after finitely many steps and exit.


BUS STATION
OIL STATION
POST OFFICE
CHILDREN'S HOSPITAL
FIRE STATIC
APRON
McDonald's
KFC
CONSTRUCTION

Parking in a digraph with a sink

General Rule

Go to the preferred vertex, if it is not available continue moving towards the sink and park at the first available spot; if no spot has been found and the sink is reached, then exit.

Parking in a digraph with a sink

General Rule

Go to the preferred vertex, if it is not available continue moving towards the sink and park at the first available spot; if no spot has been found and the sink is reached, then exit.

It is well-defined on a **rooted, labeled** tree, where each edge is oriented toward the root.

Parking in a digraph with a sink

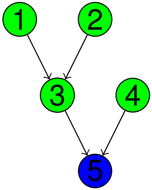
General Rule

Go to the preferred vertex, if it is not available continue moving towards the sink and park at the first available spot; if no spot has been found and the sink is reached, then exit.

It is well-defined on a **rooted, labeled** tree, where each edge is oriented toward the root.

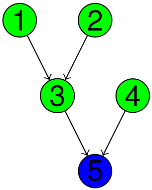
We will focus on **increasing parking functions**, or ***parking distributions***, for various family of trees.

T-parking distributions



Two representations:
(i) a monotone sequence: 11224, or
(ii) a function $f : V(T) \to \mathbb{N}$.
e.g. $f(1) = f(2) = 2, \;\; f(4) = 1, \;\; f(3) = f(5) = 0.$

T-parking distributions



Two representations:
(i) a monotone sequence: 11224, or
(ii) a function $f : V(T) \to \mathbb{N}$.
e.g. $f(1) = f(2) = 2, \;\; f(4) = 1, \;\; f(3) = f(5) = 0.$

Theorem

f is a parking distribution iff

$$\sum_{w \in T_u} f(w) \geq |T_u| \qquad \text{for all } T_u, \text{ the subtree rooted at } u, \qquad (1)$$

and the equation holds when u is the root.

Generating functions

- $p_i(T)$: The number of T-parking distributions where each vertex gets a car parked and i cars exit.
- $q_i(q;T)$: The *bump q-analogue* of the number of T-parking distributions where each vertex gets a car parked and i cars exit. In formula, $q_i(q;T)$ is a polynomial of q defined by

$$q_i(q;T) = \sum_f q^{\mathrm{bump}(f)},$$

where f ranges over all T-PDs counted by $p_i(T)$.

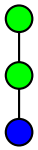
Definition

$$P_T(x) = \sum_{i\geq 0} p_i(T)x^i, \quad \text{and} \qquad Q_T(q;x) = \sum_{i\geq 0} q_i(q;T)x^i.$$

Examples

$$P_T(x) = \frac{1}{1-x}, \quad Q_T(q;x) = \frac{1}{1-qx}$$

$$P_T(x) = \frac{2-x}{(1-x)^2}, \quad Q_T(q;x) = \frac{(1+q)-q^2x}{(1-qx)(1-q^2x)}$$



$$P_T(x) = \frac{5-6x+2x^2}{(1-x)^3}$$

$$Q_T(q;x) = \frac{(1+2q+q^2+q^3)-(q^2+2q^3+2q^4+q^5)x+(q^5+q^6)x^2}{(1-qx)(1-q^2x)(1-q^3x)}$$

$$P_T(x) = \frac{3-3x+x^2}{(1-x)^3}$$

$$Q_T(q;x) = \frac{(1+2q)-(2q^2+q^3)x+q^4x^2}{(1-qx)(1-q^2x)^2}$$

Recurrences

Theorem

Let $T = T_1 \oplus T_2 \oplus \cdots \oplus T_k$. *Then*

$$P_T(x) = \frac{1}{x}\left(\frac{\prod_i P_{T_i}(x)}{1-x} - \prod_i P_{T_i}(0)\right),$$

$$Q_T(q;x) = \frac{1}{qx}\left(\frac{\prod_i Q_{T_i}(q;qx)}{1-qx} - \prod_i Q_{T_i}(0)\right).$$

Observation:

$$P_T(x) = M_T(x)/(1-x)^n$$

$$Q_T(q;x) = N_T(q;x)/\prod_{u\in T}(1-q^{d(u)+1}x)$$

for some polynomials $M_T(x)$ and $N_T(q;x)$.

Positivity

Theorem

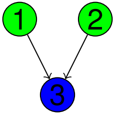
The coefficients of $M_T(-x)$ and $N_T(q;-x)$ are positive integers.

Fix a tree T and a linear order on the vertices, for a T-PD f, a vertex v is critical w.r.t. f iff for all T-PDs f' that agree with f on all the vertices less than v, $f(v)$ is maximal.

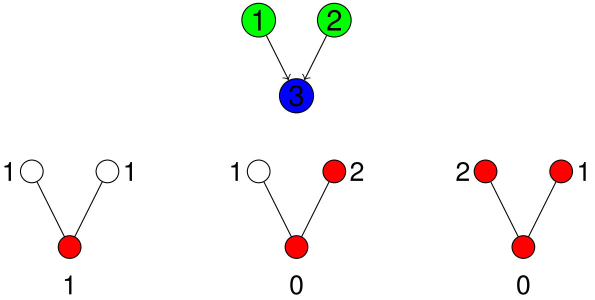
Theorem

The coefficients of $M_T(-x)$ and $N_T(q;-x)$ are determined by the numbers of critical points of T-PDs.

Examples of critical vertices



Examples of critical vertices

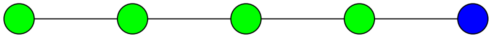


$$P_T(x) = \sum_f \frac{1}{(1-x)^{\mathrm{crit}(f)}} = \frac{1}{1-x} + \frac{1}{(1-x)^2} + \frac{1}{(1-x)^3}.$$

For the classical case, it is the maximal value in the parking function and relates to Catalan's array.

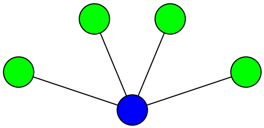
Number of T-PDs: path and star with n vertices

P_n



$$p_i(P_n) = \frac{i+2}{2n+i}\binom{2n+i}{n-1}.$$

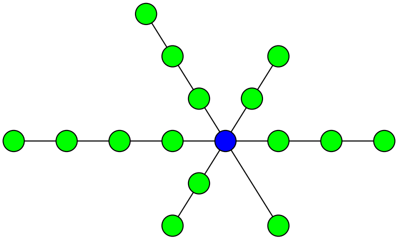
Star_n



$$p_i(\mathrm{Star}_n) = \binom{n+i}{n-1}.$$

Superstar

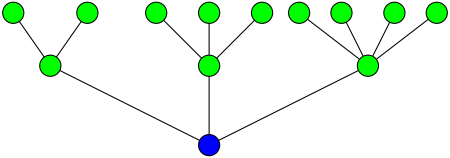
Superstar $P_{m_1} \oplus \cdots \oplus P_{m_k}$



$$p_0(T) = \left(1 + \sum_{i=1}^{k} \frac{3m_i}{m_i + 2}\right) \prod_{i=1}^{k} C_{m_i}.$$

Trees of depth 2

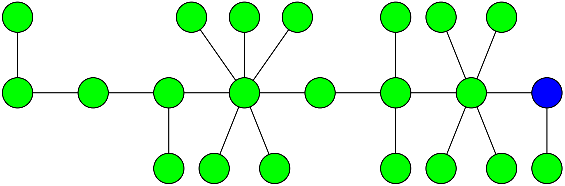
$$T = \mathrm{Star}_{m_1} \oplus \mathrm{Star}_{m_2} \oplus \cdots \oplus \mathrm{Star}_{m_k}$$



$$p_0(T) = \left(1 + \sum_{i=1}^{k} \frac{m_i + 1}{2}\right) \prod_{i=1}^{k} m_i.$$

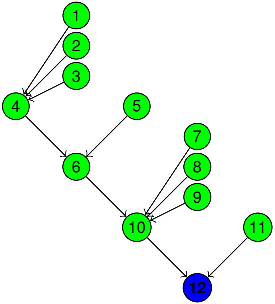
Caterpillars

$\mathrm{Cat}(a_1, a_2, \ldots, a_k)$: a path $v_1 - v_2 - \cdots v_k$ with root v_k, and each v_i is connected to an additional $a_i - 1$ leaves.



$T = \mathrm{Cat}(2, 1, 2, 6, 1, 3, 5, 2)$

Label the vertices of a caterpillar *properly*,



For a function $f : V(T) \to \mathbb{N}$, let

$$x_j = \begin{cases} f(j) - 1 & \text{if } j \text{ is a leaf,} \\ f(j) & \text{otherwise.} \end{cases}$$

Linear Equations

Then f is a T-parking distribution if and only if

$$x_1 + x_2 + \cdots + x_n = k \tag{2}$$

$$x_1 + x_2 + \cdots + x_{a_1+\cdots+a_i} \geq i \qquad \text{for } i = 1, 2, \ldots, k. \tag{3}$$

Linear Equations

Then f is a T-parking distribution if and only if

$$x_1 + x_2 + \cdots + x_n = k \tag{2}$$

$$x_1 + x_2 + \cdots + x_{a_1+\cdots+a_i} \geq i \qquad \text{for } i = 1, 2, \ldots, k. \tag{3}$$

Eqns for Cat$(4, 2, 4, 2)$

$$\begin{aligned} x_1 + x_2 + \cdots + x_{12} &= 4 \\ x_1 + x_2 + \cdots + x_4 &\geq 1 \\ x_1 + x_2 + \cdots + x_6 &\geq 2 \\ x_1 + x_2 + \cdots + x_{10} &\geq 3. \end{aligned}$$

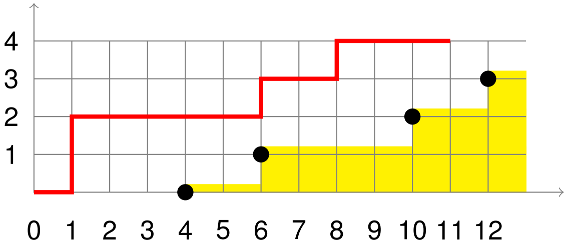
Lattice Walks

But such x_i's describe a lattice walk:

- start from the origin;
- each x_i represents a vertical step of size x_i;
- they are separated by unit horizontal steps;
- the walk ends at $(n-1, k)$.

Example of Cat(4, 2, 4, 2)

A solution: $x_2 = 2, x_7 = x_9 = 1$ and all other $x_i = 0$ for $i \leq 12$.



T-parking distributions and lattice walks

Theorem

Let $T = \mathrm{Cat}(a_1, \ldots, a_k)$ *be a caterpillar. Then the number of* T*-PDs equals the number of lattice walks from the origin to* $(n-1, k)$ *staying strictly on the left of the set of points* $\{(i-1, a_1 + \cdots + a_i) : 1 \leq i \leq k\}$. *That is,*

$$p_0(T) = \mathrm{LP}_k(a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_k),$$

T-parking distributions and lattice walks

Theorem

Let $T = \mathrm{Cat}(a_1, \ldots, a_k)$ *be a caterpillar. Then the number of* T*-PDs equals the number of lattice walks from the origin to* $(n-1, k)$ *staying strictly on the left of the set of points* $\{(i-1, a_1 + \cdots + a_i) : \ 1 \leq i \leq k\}$. *That is,*

$$p_0(T) = \mathrm{LP}_k(a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_k),$$

Good news: *lattice path enumeration with right boundaries* is well-studied.

Enumeration formulas

Corollary 1

For a caterpillar $T = \mathrm{Cat}(a_1, a_2, \ldots, a_k)$, the number of T-parking distributions is given by

$$p_0(T) = \det\left[\binom{a_1 + \cdots + a_r}{s - r + 1}\right]_{1 \le r,s \le k}. \qquad (4)$$

Corollary 2

For a t-regular caterpillar $T = \mathrm{Cat}(t+1, t, \ldots, t)$, the number of T-PDs is given by the Fuss-Catalan number

$$p_0(T) == \frac{1}{1 + t(k+1))}\binom{(1+t)(1+k)}{k+1},$$

For more involved tree, lattice walks, parking distributions and parking functions, please see the work of

Westin King