Crossings and Nestings of Two Edges in Set Partitions

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joint work with
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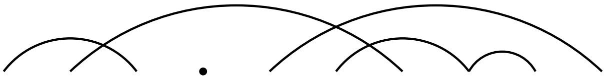
Definition of Crossings and Nestings

Graphical representation



The partition $\lambda = \{1,3\}, \{2,7\}, \{4\}, \{5,10\}, \{6,8,9\}$

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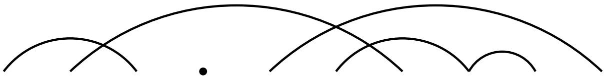

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Crossings (cr)

Nestings (ne)

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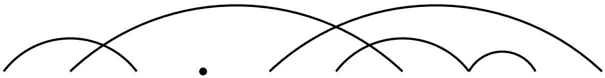


The partition $\lambda = \{1,3\},\{2,7\},\{4\},\{5,10\},\{6,8,9\}$

Crossings (cr)

Nestings (ne)

$cr(\lambda) = 3 \qquad ne(\lambda) = 2$

Previous Work

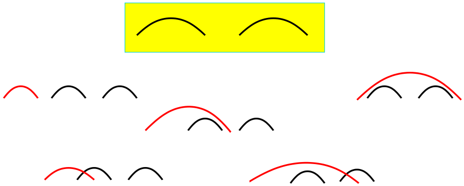
Known:

$$\sum_{\lambda \in \Pi_n} q^{cr(\lambda)} p^{ne(\lambda)} = \sum_{\lambda \in \Pi_n} q^{ne(\lambda)} p^{cr(\lambda)}$$

- Sainte-Catherine (1983, for matchings)
- Kasraoui, Zeng (2006, extended for partitions)
- Klazar (2006, studied a generating tree for matchings)

Generating rule

For matchings,



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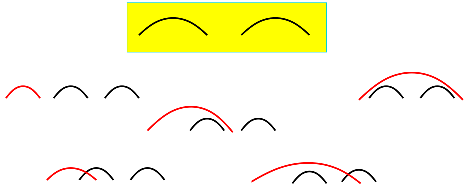
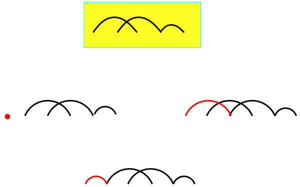
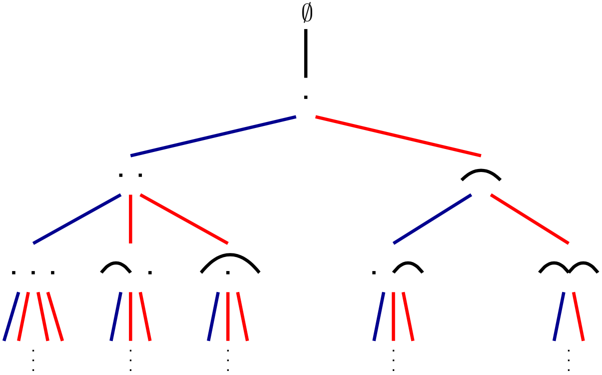


For set partitions,



The Tree $\mathcal{T}(\Pi)$ of Partitions



Questions

If $\mathcal{T}(\pi)$ and $\mathcal{T}(\lambda)$ are the subtrees of $\mathcal{T}(\Pi)$ rooted at π and λ respectively, then when can we say that:

- *cr* is equally distributed on $\mathcal{T}(\pi)$ and $\mathcal{T}(\lambda)$
- *ne* is equally distributed on $\mathcal{T}(\pi)$ and $\mathcal{T}(\lambda)$

Questions

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- *cr* is equally distributed on $\mathcal{T}(\pi)$ and $\mathcal{T}(\lambda)$
- *ne* is equally distributed on $\mathcal{T}(\pi)$ and $\mathcal{T}(\lambda)$
- (*cr*, *ne*) is equally distributed on $\mathcal{T}(\pi)$ and $\mathcal{T}(\lambda)$
- the distribution of (*cr*, *ne*) on $\mathcal{T}(\pi)$ is equal to the distribution of (*ne*, *cr*) on $\mathcal{T}(\lambda)$

The Statistic $s_{\alpha,\beta}$

Let G be an Abelian group and α, $\beta \in G$. Define

$$s_{\alpha,\beta}(\lambda) = cr(\lambda)\alpha + ne(\lambda)\beta$$

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Example

- If $G = \mathbb{Z}$, $\alpha = 1$, $\beta = 0$ then $s_{\alpha,\beta} = cr$ and $s_{\beta,\alpha} = ne$
- If $G = \mathbb{Z} \oplus \mathbb{Z}$, $\alpha = (1, 0)$, $\beta = (0, 1)$ then $s_{\alpha,\beta} = (cr, ne)$ and $s_{\beta,\alpha} = (ne, cr)$

Main Theorem

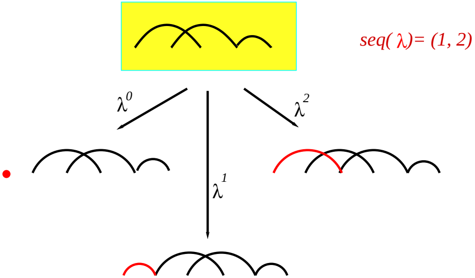
Theorem

- *The statistic $s_{\alpha,\beta}$ is equally distributed on all the corresponding levels of the subtrees $\mathcal{T}(\pi)$ and $\mathcal{T}(\lambda)$ if and only if it is equally distributed on the first two levels.*
- *Same result holds when comparing the distributions of $s_{\alpha,\beta}$ on $\mathcal{T}(\pi)$ and $s_{\beta,\alpha}$ on $\mathcal{T}(\lambda)$.*

Outline of Proof

λ: a partition with k blocks

- Children of λ: $\lambda^0, \lambda^1, \ldots, \lambda^k$
- Associate to λ a sequence $(x_1, x_2, \ldots, x_k)$ where $x_i = s_{\alpha,\beta}(\lambda^i)$



Proof, continued

- Recurrence for the sequences:
  If $s_{\alpha,\beta}(\lambda) = (x_1, x_2, \ldots, x_k)$, then

$$\begin{aligned} s_{\alpha,\beta}(\lambda^0) &= (x_1, x_1, x_2, \ldots, x_k) \\ s_{\alpha,\beta}(\lambda^i) &= (x_i, x_1 + A, \ldots, x_{i-1} + A, x_{i+1} + B, \ldots, x_k + B), \end{aligned}$$

  where $A = x_i - x_1 + \alpha$, $B = x_i - x_1 + \beta$.

Note that $s_{\alpha,\beta}(\lambda)$ on the first two level of $\mathcal{T}(\lambda)$ are

$$\{x_1\} \text{ and } \{x_1, x_1, x_2, \ldots, x_k\}$$

Lemmas

For $\gamma \in G$ and nonnegative integer r:

$$f_\gamma^r(x_1x_2\dots x_k) := \{x_{a_1}+\cdots+x_{a_r}-(r-1)x_1+\gamma : 1 < a_1 < \cdots < a_r \le k\}$$

$$f_0^0(x_1x_2\dots x_k) = \{x_1\}$$

$$f_0^1(x_1x_2\dots x_k) = \{x_2,\dots,x_k\}$$

Lemma 1

If $x_1 \dots x_k$ and $y_1 \dots y_k$ are two sequences of elements in G such that $x_1 = y_1$ and $\{x_2,\dots,x_k\} = \{y_2,\dots,y_k\}$ as multisets then $f_\gamma^r(x_1\dots x_k) = f_\gamma^r(y_1\dots y_k)$ for all r and γ.

Lemma 2

If X and Y are two multisets of sequences over elements in G such that $f_\gamma^r(X) = f_\gamma^r(Y)$ for all γ and r then

$$f_\gamma^r(\text{children of sequences in } X) = f_\gamma^r(\text{children of sequences in } Y)$$

for all γ and r.

Proof Ends

Use induction to complete the proof.

$$X = seq(\lambda), \qquad Y = seq(pi)$$

- $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$ agree on the first two level means: $f_0^0(X) = f_0^0(Y)$ and $f_0^1(X) = f_0^1(Y)$.
- Using Lemma 1, $f_\gamma^r(X) = f_\gamma^r(Y)$ for all r and γ.
- Using Lemma 2 and induction on m,

$$f_\gamma^r(\text{sequences, level } m, \mathcal{T}(\lambda)) = f_\gamma^r(\text{sequences, level } m, \mathcal{T}(\pi))$$

- Take $\gamma = r = 0$.

Theorem can be refined to fix the number of blocks.

Corollary

(Kasraoui,Zeng,2006)

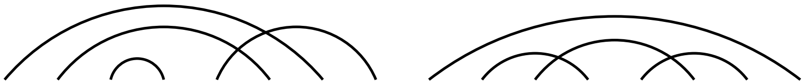
$$\sum_{\pi \in \Pi_n} p^{cr(\pi)} q^{ne(\pi)} = \sum_{\pi \in \Pi_n} p^{ne(\pi)} q^{cr(\pi)}$$

Proof: $G = (\mathbb{Z} \oplus \mathbb{Z}, +)$, $\alpha = (1, 0)$, $\beta = (0, 1)$, and $\lambda = \pi = \{\{1\}\}$. The result follows from the second part of the Theorem because the partitions of 1 and 2 elements have 0 crossings and nestings.

Example

Let $\lambda = \{\{1,7\},\{2,6\},\{3,4\},\{5,8\}\}$ and $\pi = \{\{1,8\},\{2,4\},\{3,6\},\{5,7\}\}$. Then

$$\begin{aligned} & \#\{\gamma \text{ on } [n] : cr(\gamma) = m, ne(\gamma) = l, \gamma|_{\text{last eight points}} \cong \lambda\} \\ = \; & \#\{\gamma \text{ on } [n] : cr(\gamma) = m, ne(\gamma) = l, \gamma|_{\text{last eight points}} \cong \pi\}. \end{aligned}$$



Proof: Again set $G = (\mathbb{Z} \oplus \mathbb{Z}, +)$, $\alpha = (1,0)$ and $\beta = (0,1)$. Then $s_{\alpha,\beta} = (cr, ne)$. The claim follows from part (a) of the Theorem since

$$s_{\alpha,\beta}(\lambda) = (2,3) = s_{\alpha,\beta}(\pi)$$

and

$$s_{\alpha,\beta}(\mathcal{T}(\lambda,1)) = \{(2,3),(2,3),(3,3),(4,3),(4,4)\} = s_{\alpha,\beta}(\mathcal{T}(\pi,1)).$$

Crossing/nesting Similarity Classes

- For $\lambda, \pi \in \Pi_n$ we say $\lambda \sim_{cr} \pi$ if and only if *cr* is equally distributed on $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$.
- For $\lambda, \pi \in \Pi_n$ we say $\lambda \sim_{ne} \pi$ if and only if *ne* is equally distributed on $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$.

Charlier diagram

Let $\lambda = \{\{1,3,4\},\{2,6,7\},\{5\}\}$.

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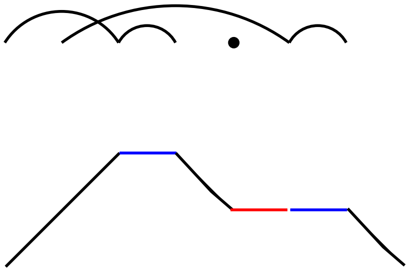

Charlier diagram

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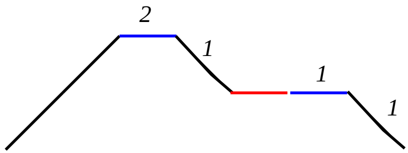
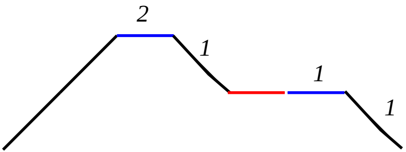


Read information from Charlier diagram



- # blocks = #(up and red steps).
- $cr(\lambda) = \sum(w_i - 1)$, $ne(\lambda) = \sum(h_i - w_i)$.
- $\{cr(\lambda^i) - cr(\lambda) : i \geq 1\} = \{$height of up and red steps$\}$.
- $\{ne(\lambda^i) - ne(\lambda) : i \geq 1\} = \{p_i - i : i = 1, ..., k\}$ where p_i's are positions of up and red steps.

Crossing-similarity classes

Theorem

Let $n \geq k \geq 1$ *and* $m = \min\{n-k, k-1\}$. *Then*

$$|\Pi_{n,k}/\sim_{cr}| = \sum_{l=0}^{m} \binom{k-1}{l} \left[(n-k-1)l - \frac{l(l-1)}{2} + 1\right]$$

In particular, if $n \geq 2k-1$,

$$\left|\Pi_{n,k}/\sim_{cr}\right| = (n-k-1)(k-1)2^{k-2} + 2^{k-1} - (k-1)(k-2)2^{k-4}$$

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$|\Pi_{2k,k}| > (2k-1)!! \approx \sqrt{2}\left(\frac{2k}{e}\right)^k$ while $\left|\Pi_{2k,k}/\sim_{cr}\right| \approx 3k^2 2^{k-4}$

Nesting-similarity classes

Theorem

Let $n \geq k \geq 1$. *If* $f_{n,k} = |\Pi_{n,k}/\sim_{ne}|$ *then*

$$f_{n,1} = 1,$$

$$f_{n,k} = \sum_{r=k-1}^{n-1} f_{r,k-1} + (k-1)\binom{n-2}{k}, \qquad k \geq 2$$

Corollary

$$|\Pi_1/\sim_{ne}| = 1, \qquad |\Pi_2/\sim_{ne}| = 2$$

$$|\Pi_n/\sim_{ne}| = 2^{n-5}(n^2 - 5n + 22), \qquad n \geq 3$$

Generating Function for Crossing and Nestings

If π has one block:

$$\sum_{l\geq 0} \sum_{\lambda\in\mathcal{T}(\pi,l)} q^{cr(\lambda)} p^{ne(\lambda)} z^l =$$

$$= \cfrac{1}{1-([1]_{q,p}+1)z-\cfrac{[1]_{q,p}z^2}{1-([2]_{q,p}+1)z-\cfrac{[2]_{q,p}z^2}{\ddots}}}$$

where $[n]_{q,p} := \frac{q^n-p^n}{q-p}$

Thank you!

Preprint can be found at:

arXiv:0710.1816
or
www.math.tamu.edu/∼cyan/papers.html