

# MATH 302 Discrete Mathematics

## Solution for Extra-credit Assignment 2.

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1. Let  $f_n$  be the  $n$ -th Fibonacci number. Prove that  $f_0f_1 + f_1f_2 + \cdots + f_{2n-1}f_{2n} = f_{2n}^2$  when  $n$  is a positive integer.

**Proof.** Use induction. For  $n = 1$ ,  $LHS = f_0f_1 + f_1f_2 = 1$ ,  $RHS = f_2^2 = 1$ . The equation holds.

Assume that for  $n = k$ , the equation holds. We shall prove it for  $n = k + 1$ . Note that

$$\begin{aligned} LHS &= f_0f_1 + f_1f_2 + \cdots + f_{2k-1}f_{2k} + f_{2k}f_{2k+1} + f_{2k+1}f_{2k+2} \\ &= f_{2k}^2 + f_{2k}f_{2k+1} + f_{2k+1}f_{2k+2} \\ &= f_{2k}(f_{2k} + f_{2k+1}) + f_{2k+1}f_{2k+2} \\ &= f_{2k}f_{2k+2} + f_{2k+1}f_{2k+2} \\ &= f_{2k+2}(f_{2k} + f_{2k+1}) \\ &= f_{2k+2}^2. \end{aligned}$$

This finishes the induction .

2. Find the solution to the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with  $a_0 = 2$  and  $a_1 = 7$ .

**Solution.** The characteristic equation is  $q^2 - q - 2 = 0$ , where the two roots are 2 and  $-1$ . Hence  $a_n = C2^n + D(-1)^n$ , where  $C$  and  $D$  are some constants.

By the initial values  $a_0 = 2$  and  $a_1 = 7$ , we have

$$\begin{aligned} C + D &= 2 \\ 2C - D &= 7. \end{aligned}$$

Solving the system of linear equations, we get  $C = 3$ ,  $D = -1$ . Hence

$$a_n = 3 \cdot 2^n - (-1)^n.$$

3. In how many ways can one choose 10 integers  $a_1, a_2, \dots, a_{10}$  from the range  $[1, 100]$  such that for any pair of the chosen numbers, the difference is at least 2?

**Solution.** We need to count the number of ways to choose 10 integers  $a_1 < a_2 < \cdots < a_{10}$  from 1 to 100 such that  $a_{i+1} - a_i \geq 2$ . Now let  $b_i = a_i - (i - 1)$ . Then the sequences  $(a_1, \dots, a_{10})$  are in one-to-one correspondence with the sequences  $(b_1, \dots, b_{10})$  where  $1 \leq b_1 < b_2 < \cdots < b_{10} \leq 91$ . Hence the answer is  $\binom{91}{10}$ .

4. Let  $(x_i, y_i)$ ,  $i = 1, 2, 3, 4, 5$  be a set of five distinct points with integer coordinates in the  $xy$  plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

**Proof.** The middle point of  $(x_i, y_i)$  and  $(x_j, y_j)$  is  $(\frac{x_i+x_j}{2}, \frac{y_i+y_j}{2})$ . The middle point has integer coordinates if and only if  $(x_i, y_i)$  has the same parity of  $(x_j, y_j)$ . There are only four possibilities for the parity of a pair of integers, namely, (even, even), (even, odd), (odd, even), and (odd, odd). For five distinct integers, by Pigeonhole Principle, there are at least two points with the same parity. The middle point of these two has integer coordinates.

5. For  $n \in \mathbb{Z}^+$ , consider the following sum

$$\sum_{i=1}^n \frac{1}{(2i-1)(2i+1)}.$$

make a conjecture about the value of the sum, and prove your conjecture.

**Solution.** One computes that for  $n = 1, 2, 3$ , the sum is  $1/3, 2/5$ , and  $3/7$ . Hence conjecture:

$$\sum_{i=1}^n \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}.$$

One can prove it by induction. Another way is the notice that

$$\frac{1}{(2i-1)(2i+1)} = \frac{1}{2} \left( \frac{1}{2i-1} - \frac{1}{2i+1} \right).$$

Therefore

$$\begin{aligned} \sum_{i=1}^n \frac{1}{(2i-1)(2i+1)} &= \frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2i-1} - \frac{1}{2i+1} \right) \\ &= \frac{1}{2} \left( 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2n+1} \right) \\ &= \frac{1}{2} \left( 1 - \frac{1}{2n+1} \right) \\ &= \frac{n}{2n+1}. \end{aligned}$$