MATH 302. Discrete Mathematics
Extra-credit Assignment 3. Solution

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1. How many positive perfect squares less than 10^6 are multiples of 24?

   Solution.n^2 is a multiple of $24 = 2^3 \times 3$ if and only if n is a multiple of 12. $1 \leq n^2 \leq 10^6$ means $1 \leq n \leq 1000$. So the answer is $\lfloor \frac{1000}{12} \rfloor = 83$.

2. Prove that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

   *Proof.*Suppose we want to choose some representatives from a group of n people, and identity one of the chosen person to be the leader. We can do it by choosing the leader first, (n ways), and then choose a group of other representatives, (in 2^{n-1} ways.) Hence the total number of ways to do the job is $n2^{n-1}$.

   On the other hand, we can do it by choosing k representatives first, (in $\binom{n}{k}$ ways), then pick up a leader from those representatives (there are k ways here). Summary over k we get the total number of ways to choose the group with leader is $\sum_{k=1}^{n} k\binom{n}{k}$. Hence the equation follows.

3. How many ways are there to travel in xyz space from the origin $(0, 0, 0)$ to the point $(100, 200, 300)$ by taking steps one unit in the positive x direction, one unit in the positive y direction, or one unit in the positive z direction? (Moving in the negative x, y or z direction is prohibited, so that no backtracking is allowed.)

   *Solution.*This is equivalent to the following problem: arranging 100 x, 200 y and 300 z in a row, how many different arrangement can one have.

   The answer is $\binom{600}{100,200,300}$.

4. Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

   *Proof.*Pick one person v, and consider the relations between v and others. Among the remaining five, there must be three people who are all friends of v, or all enemies of v. Without loss of generality, assume u, w, x are friends of v. Now consider the pairs $(u, w), (u, x)$ and (w, x). If any of them is friend, then together with v we have three mutual friends. Otherwise, all three pairs are enemies. Then three mutual enemies.

5. Let A be a set with 100 elements. How many relations on A that are reflexive and anti-symmetric?

   *Solution.*A relation is reflexive means the diagonal elements are all 1. Anti-symmetry means for each off-diagonal pair, there are three choices. So the answer if $3^{(n^2-n)/2} = 3^{4950}$.