Review for the Final Test

Materials covered in the third test: Chapter 6.1, 6.3–6.5, Counting, Permutations and Combinations; Chapter 8.1-3, 8.5-8.6, recurrence relations, the Masters Theorem, and Inclusion-Exclusion; Chapter 9.1–9.4 relations.

1 Chapter 8. Recurrence relations and Masters Theorem

Need to know: How to solve first and second order homogeneous recurrence relations, and nonhomogeneous ones for the first order. The statement of Masters Theorem.

Sample problems:

1. Find the asymptotic bound for the following function $f$:
   (1) $f(n) = 2f(n/2) + n$
   (2) $f(n) = 2f(n/2) + n^2$
   (3) $f(n) = 2f(n/2) + \log n$
   (4) $f(n) = 4f(n/3) + n \log n$

2. Find the general solution of the following recurrence relations.
   (1) $a_n = 3a_{n-1}$
   (2) $a_n = a_{n-1} + 2a_{n-2}$
   (3) $a_n = 4a_{n-2}$
   (4) $a_n = 10a_{n-1} - 25a_{n-2}$

3. Find the general solution of the following recurrence relations.
   (1) $a_n = 3a_{n-1} + 4$
   (2) $a_n = 3a_{n-1} - 5n$
   (3) $a_n = 4a_{n-1} + 3 \cdot 2^n$
   (4) $a_n = 4a_{n-1} - 4^n$

4. Solve the recurrence with initial conditions.
   (1) $a_n = 3a_{n-1} + 2n$, and $a_0 = 4$.
   (2) $a_n = 5a_{n-1} + 6a_{n-2}$ and $a_0 = 4$, $a_1 = 12$.

2 Chapter 6.Counting

Need to know: The product rule, the sum rule; permutations and combinations, binomial theorem, and generalized permutation and combinations

Sample problems.

1. (1) The number of binary strings of length $n$?
   (2) If $A$ and $B$ are finite sets, how many functions are there from $A$ to $B$? How many are one-to-one functions?
   (3) If $A$ and $B$ are finite sets, what is $|A \times B|$? How many relations are there from $A$ to $B$?
2. How many permutations are there for A, B, C, D, E, F, G, H? How many contain the segment ABC?

3. How many ways to select r people from a team of n? How many ways can you do it if the team caption must be selected?

4. Number of ways to choose 10 integers $a_1, a_2, \ldots, a_{10}$ from $\{1, 2, \ldots, 100\}$ such that $a_1 < a_2 < \cdots < a_{10}$?

5. Number of ways to choose 10 integers $a_1, a_2, \ldots, a_{10}$ from $\{1, 2, \ldots, 100\}$ such that $a_1 \leq a_2 \leq \cdots \leq a_{10}$?

6. Number of binary strings of length n with exactly k 1’s?

7. Number of binary strings made of k 1’s and r 0’s?

8. Number of binary strings of length n with an even number of 1’s?

9. The coefficient of the term $x^{20}y^{25}$ in $(3x - 2y)^{45}$?

10. Evaluation $\sum_{k=0}^{n} \binom{n}{k} 2^k =$

11. Evaluation $\sum_{k=0}^{n} \binom{n}{k} 2^{2k} =$

12. The number of strings can be made by reordering the letters of the word "SUCCES"?

13. The number of 8-digit positive integers made by using 1, 1, 2, 2, 2, 3, 3, 3?

14. The number of ways to put n different markers into k distinguishable boxes?

15. The number of ways to put n identical markers into k distinguishable boxes?

16. The number of ways to put n identical markers into k distinguishable boxes so that no box is empty?

17. The number of integer solutions for the equation $x_1 + x_2 + \cdots + x_r = n$ with $x_i \geq 0$?

18. The number of integer solutions for the equation $x_1 + x_2 + x_3 + x_4 = 100$ with $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$ and $x_4 \geq 4$?

3 Chapter 9. Relations

Need to know: Definitions of relations, reflexive, symmetric, antisymmetric, and transitive, the representation of relations by matrix, directed-graphs, reflexive closure, symmetric closure, and transitive closure.

Sample problems.

1. Let $A$ be a set with $n$ elements. The number of relations on $A$, the number of relations on $A$ that are reflexive, symmetric, or antisymmetric? (Any combination of these three properties).
2. Decide whether the following relations are reflexive, symmetric, anti-symmetric, or transitive.
   (1) The empty relation $\emptyset$, the full relation $A \times A$, and the identity relation $I$
   (2) $A = \mathbb{N}$, $(a, b) \in R$ iff $a < b$.
   (3) $A = \mathbb{N}$, $(a, b) \in R$ iff $a \leq b$.
   (4) $A = \mathbb{R}$, $(a, b) \in R$ iff $a - b$ is an integer.
   (5) $A = P(\mathbb{N})$ (all the subsets of $\mathbb{N}$), $(a, b) \in R$ iff $a$ is a subset of $b$.

3. Find the reflexive closure, symmetric closure, and transitive closure of the following relation
   on $\mathbb{N}$: $(a, b) \in R$ iff $b - a = 1$.
   Answer the above question with $(a, b) \in R$ iff $b - a = 2$.

4. Decide whether the following relations are equivalence relation, partial order, or none of the two.
   (1) Let $A$ be the set of binary strings. For two strings $a$ and $b$, $(a, b) \in R$ if and only if
       the length of $a$ is the same as the length of $b$.
   (2) Let $A$ be the set of all continuous functions defined on $R$. $(f, g) \in R$ if and only if
       $f(0) = g(0)$.
   (3) Let $A$ be the set of all continuous functions defined on $R$. $(f, g) \in R$ if and only if
       $f(0) \leq g(0)$.
   (4) Let $A$ be the set of all continuous functions defined on $R$. $(f, g) \in R$ if and only if
       $f(x) \leq g(x)$ for all $x$.
   (5) Let $A$ be the set of all continuous functions defined on $R$. $(f, g) \in R$ if and only if
       $f(x) \leq g(x)$ for some $x$.
   (6) $A = \mathbb{R}$. Let $f(x)$ be a function from $\mathbb{R}$ to $\mathbb{R}$. For two real numbers $a$ and $b$, $(a, b) \in R$ if
       and only if $f(a) = f(b)$.

5. Define a relation $R$ on the set of binary strings of length 3 by setting $(a, b) \in R$ if and only if
   the first character of $a$ is the same as the first character of $b$. check that $R$ is an equivalence
   relation.

For review: please also look at the homework problems.

Answers

Chapter 8:
1. (1) $f(n) = \Theta(n \log n)$; (2) $f(n) = \Theta(n^2)$. (3) $f(n) = \theta(n)$, (4) $f(n) = \Theta(n \log_3 4)$.
2. (1) $a_n = \alpha 3^n$, (2) $a_n = \alpha 2^n + \beta(-1)^n$. (3) $a_n = \alpha 2^n + \beta(-2)^n$. (4) $a_n = \alpha 5^n + \beta n 5^n$.
3. (1) $a_n = \alpha 3^n - 2$. (2) $a_n = \alpha 3^n + \frac{5}{2} n + \frac{15}{2}$. (3) $a_n = \alpha 4^n - 2 \cdot 2^n$. (4) $a_n = \alpha 4^n - n 4^n$.
4. (1) The general solution of $a_n$ is $a_n = \alpha 3^n - n - \frac{3}{2}$. Using $a_0 = 4$, one gets $\alpha = \frac{11}{2}$.
   (2) The general solution is $a_n = \alpha 6^n + \beta(-1)^n$. Using $a_0 = 4$ and $a_1 = 12$, one gets $\alpha = \frac{16}{7}$ and
   $\beta = \frac{12}{7}$.

Chapter 5.
1. (1) $2^n$; (2) $|B|^{|A|}$, $P(|B|, |A|)$. (3) $|A||B|$, $2^{|A||B|}$.
2. $8!$, $6!$.
3. \( \binom{n}{r} \), \( \binom{n-1}{r-1} \).
4. \( \binom{10}{0} \).
5. This is the combination with repetition: choose 10 out of 100, allowing repetition. \( \binom{109}{10} \).
6. \( \binom{n}{k} \).
7. \( \binom{k+r}{k} \).
8. \( 2^n-1 \).
9. \( \left( \begin{array}{c} 45 \\ 25 \end{array} \right) 3^{20} (-2)^{25} \).
10. \( (2+1)^n = 3^n \).
11. Note that \( 2^n k = 4^n k \). Answer is \( \left( \begin{array}{c} 4+1 \\ n \end{array} \right) = 5^n \).
12. \( \left( \begin{array}{c} 7 \\ 1, 1, 2, 3 \end{array} \right) \).
13. \( \left( \begin{array}{c} 8 \\ 2, 3, 3 \end{array} \right) \).
14. Every marker has \( k \) choices. \( k^n \).
15. Let \( x_i \) be the number of markers in the \( k \)-th box, Then \( x_1 + x_2 + \cdots + x_k = n \), and \( x_i \geq 0 \). So the answer is \( \left( \begin{array}{c} n+k-1 \\ n \end{array} \right) \).
16. It is the number of integer solutions of \( x_1 + \cdots + x_k = n \) and \( x_i \geq 1 \). So the answer is \( \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right) \).
17. \( \left( \begin{array}{c} n+r-1 \\ n \end{array} \right) \).
18. Let \( y_i = x_i - i \). So \( y_1 + y_2 + y_3 + y_4 = 90 \). Answer is \( \left( \begin{array}{c} 93 \\ 3 \end{array} \right) \).

Chapter 8.
1. Number of relations: \( 2^{n^2} \). Number of relations that are reflexive: \( 2^{n^2-n} \). Number of relations that are symmetric: \( 2^{(n^2+n)/2} \). Number of relations that are anti-symmetric: \( 2^n 3^{(n^2-n)/2} \) Number of relations that are reflexive and symmetric: \( 2^{(n^2-n)/2} \) Number of relations that are reflexive and antisymmetric: \( 3^{(n^2-n)/2} \) Number of relations that are symmetric and antisymmetric: \( 2^n \). Number of relations that are reflexive, symmetric and anti-symmetric: 1.
2. (1) \( \emptyset \): symm, antisymm; \( A \times A \): reflexive, symmetric, transitive, not anti-symmetric unless \( A \leq 1 \); \( I \): ref, symm, anti-symm, transitive. (2) anti-symmetric. transitive. (3) ref, anti-sym, trans. (4) ref, symm, trans. (5) ref, anti-sym, trans.
3. For \( (a,b) \in R \) iff \( b-a = 1 \): reflexive closure: \( \{(a,b) : b-a = 0 \text{ or } 1\} \), symmetric closure: \( \{(a,b) : |b-a| = 1\} \), transitive closure: \( \{(a,b) : b > a\} \).
For \( (a,b) \in R \) iff \( b-a = 2 \): reflexive closure: \( \{(a,b) : b-a = 0 \text{ or } 2\} \), symmetric closure: \( \{(a,b) : |b-a| = 2\} \), transitive closure: \( \{(a,b) : b-a \text{ is even}\} \).
4. (1) equivalence relation (2) equivalence relation (3) partial order (4) partial order (5) none of above (6) equivalence relation.