A basic problem in Combinatorics is to count functions between two sets. Let \( N \) and \( X \) be finite sets with \(|N| = n\) and \(|X| = x\). We wish to count the number of functions \( f : N \to X \) subject to certain restrictions. There will be three restrictions on the functions themselves and four restrictions on when we consider two functions to be the same. This gives a total of twelve counting problems, and their solution is called the *Twelvefold way*.

The three restrictions on the functions \( f : N \to X \) are the following:

1. \( f \) is arbitrary (no restriction).
2. \( f \) is injective, (one-to-one). That is, \( f(a) = f(b) \) implies \( a = b \).
3. \( f \) is surjection, (onto). That is, for any \( x \in X \), there are some \( a \in N \) such that \( f(a) = x \).

The four interpretations as to when two functions are the same come about from regarding the elements of \( N \) and \( X \) as “distinguishable” or “indistinguishable”. Think of \( N \) as a set of balls and \( X \) as a set of boxes. A function \( f : N \to X \) consists of placing each ball into some box. If we can tell the balls apart, then the elements of \( N \) are called *distinguishable*, otherwise *indistinguishable*. Similarly, if we can tell the boxes apart, then the elements of \( X \) are called *distinguishable*, otherwise *indistinguishable*. For example, suppose \( N = \{1, 2, 3\} \), and \( X = \{a, b, c, d\} \), and define functions \( f, g, h, i : N \to X \) by

\[
\begin{align*}
f(1) &= f(2) = a, & f(3) &= b, \\
g(1) &= g(3) = a, & g(2) &= b, \\
h(1) &= h(2) = b, & h(3) &= d, \\
i(2) &= i(3) = b, & i(1) &= c.
\end{align*}
\]

If the elements of both \( N \) and \( X \) are distinguishable, the functions have the pictures shown in Figure 1. All four pictures are different, and the four functions are different.
Now suppose that the elements of $N$ (but not $X$) are indistinguishable. This corresponds to erasing the labels on the balls. The pictures for $f$ and $g$ both become the following one. So now $f$ and $g$ are equivalent. However, $f$, $h$ and $i$ remain inequivalent.

If the elements of $X$ (but not $N$) are indistinguishable, then we erase the labels of the boxes. Thus $f$ and $h$ both have the picture shown in figure 3. Hence $f$ and $h$ are equivalent, but $f$, $g$, $i$ are inequivalent.

If the elements of both $N$ and $X$ are indistinguishable, then all four functions have the picture shown in Figure 4, so all four are equivalent.
Figure 4: All four are equivalent when both $N$ and $X$ are indistinguishable.

In this project, you are required to compute the number of functions in each case.

1. Compute the entries in the following table for $N = \{1, 2, 3\}$ and $X = \{a, b, c, d\}$.

2. Compute the entries in the following table for $N = \{1, 2, 3, 4\}$ and $X = \{a, b, c\}$.

3. Derive a formula for each entry in the following table. Note that in some cases, the answer may not have a nice closed formula.

   For each answer you get, give some explanation. You can use other research tools, such as books, papers, and websites.

<table>
<thead>
<tr>
<th>Elements of N</th>
<th>Elements of X</th>
<th>Any $f$</th>
<th>Injective $f$</th>
<th>Surjective $f$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>dist.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>dist.</td>
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<td>indist.</td>
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