MATH 431-200. Structures and Methods in Combinatorics
Optional Project: The Twelvefold Way.
Due on Apr. 22, 2008

A basic problem in Combinatorics is to count functions between two sets. Let N and X be finite sets with $|N| = n$ and $|X| = x$. We wish to count the number of functions $f : N \to X$ subject to certain restrictions. There will be three restrictions on the functions themselves and four restrictions on when we consider two functions to be the same. This gives a total of twelve counting problems, and their solution is called the *Twelvefold way*.

The three restrictions on the functions $f : N \to X$ are the following:

1. f is arbitrary (no restriction).
2. f is injective, (one-to-one). That is, $f(a) = f(b)$ implies $a = b$.
3. f is surjection, (onto). That is, for any $x \in X$, there are some $a \in N$ such that $f(a) = x$.

The four interpretations as to when two functions are the same come about from regarding the elements of N and X as "distinguishable" or "indistinguishable". Think of N as a set of balls and X as a set of boxes. A function $f : N \to X$ consists of placing each ball into some box. If we can tell the balls apart, then the elements of N are called *distinguishable*, otherwise *indistinguishable*. Similarly, if we can tell the boxes apart, then the elements of X are called *distinguishable*, otherwise *indistinguishable*. For example, suppose $N = \{1, 2, 3\}$, and $X = \{a, b, c, d\}$, and define functions $f, g, h, i : N \to X$ by

$$\begin{aligned} f(1) = f(2) &= a, & f(3) &= b, \\ g(1) = g(3) &= a, & g(2) &= b, \\ h(1) = h(2) &= b, & h(3) &= d, \\ i(2) = i(3) &= b, & i(1) &= c. \end{aligned}$$

If the elements of both N and X are distinguishable, the functions have the pictures shown in Figure 1. All four pictures are different, and the four functions are different.

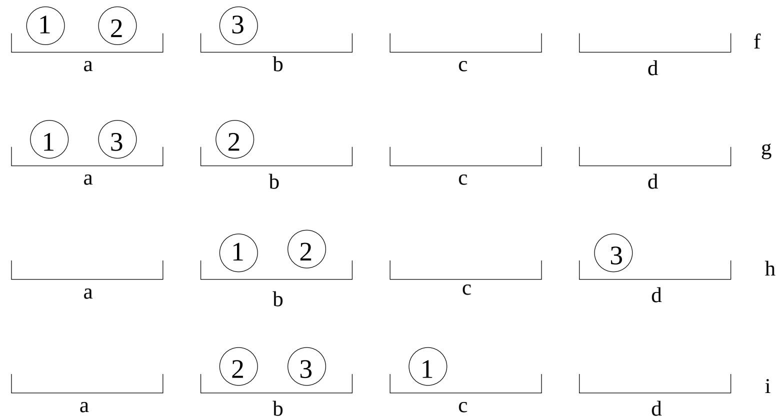

Figure 1: All four functions

Now suppose that the elements of N (but not X) are indistinguishable. This correspondents to erasing the labels on the balls. The pictures for f and g both become the following one. So now f and g are equivalent. However, f, h and i remain inequivalent.

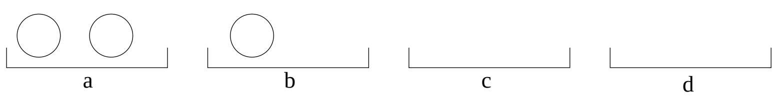

Figure 2: Functions f and g are equivalent when N is indistinguishable.

If the elements of X (but not N) are indistinguishable, then we erase the labels of the boxes. Thus f and h both have the picture shown in figure 3. Hence f and h are equivalent, but f, g, i are inequivalent.

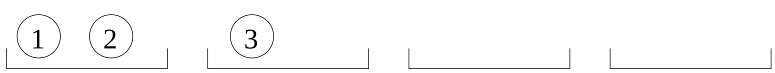

Figure 3: Functions f and h are equivalent with X is indistinguishable.

If the elements of both N and X are indistinguishable, then all four functions have the picture shown in Figure 4, so all four are equivalent.

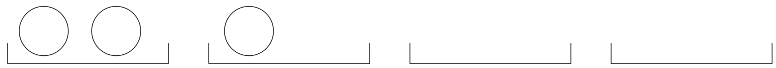

Figure 4: Al four are equivalent when both N and X are indistinguishable.

In this project, you are required to compute the number of functions in each case.

1. Compute the entries in the following table for $N = \{1, 2, 3\}$ and $X = \{a, b, c, d\}$.
2. Compute the entries in the following table for $N = \{1, 2, 3, 4\}$ and $X = \{a, b, c\}$.
3. Derive a formula for each entry in the following table. Note that in some cases, the answer may not has a nice closed formula.

   For each answer you get, give some explanation. You can use other research tools, such as books, papers, and websites.

Elements of N	Elements of X	Any f	Injective f	Surjective f
dist	dist.			
indist.	dist.			
dist.	indist.			
indist.	indist.			