MATH 630. Combinatorics

Assignment 5. Due on Wednesday, October 24, 2018

For all questions, please write your argument in full details. Discussion with your classmates or checking on reference books are all right. But every student must write his/her OWN solutions.

The problems are from the textbook *A Course in Enumeration* by Martin Aigner. Springer 2007.
Page 110: 3.31, 3.40, 3.41
Page 122: 3.44, 3.46, 3.50, 3.63.

Supplement problem:

1. We want to find the sum of the first n terms in the binomial expansion of

$$\left(1-\frac{1}{2}\right)^{-n} = 1 + \frac{1}{2}n + \frac{1}{4}\binom{n+1}{2} + \cdots$$

For instance, when $n = 3$, we get $1 + \frac{3}{2} + \frac{6}{4} = 4$.

   (a) Prove that the sum of the first n terms in the above formula is the coefficient of x^{n-1} in the formal power series of

$$F(x) = (1 - \frac{1}{2}x)^{-n}(1-x)^{-1}.$$

   (b) Apply the "strong form" of the Lagrange Inversion Formula: Let $H(x) = xG(x)$, $F(x)$ be the compositional inverse of $H(x)$, and $T(x)$ is a formal power series. Then

$$n[x^n]T(F(x)) = [x^{n-1}]T'(x)(G(x))^{-n}.$$

   In the above formula, assume $T'(x) = (1-x)^{-1}$ and $G(x) = 1 - \frac{1}{2}x$. Find $T(x)$, $H(x)$, and $F(x) = H^{<-1>}(x)$.

   (c) Compute the function $T(F(x))$.

   (d) Find $n[x^n]T(F(x))$.

2. Assume R is a set of restrictions. Let $L((a,b) \to (c,d) \mid R)$ be the set of lattice paths in plane from (a,b) to (c,d) that consist of steps $(1,0)$ and $(0,1)$, and satisfy the restrictions R. For example,

$$L((0,0) \to (n,n) \mid x \geq y)$$

is the set of all sub-diagonal lattice paths from $(0,0)$ to (n,n), which is counted by the n-th Catalan number.

Let μ and k be positive integers. To count the number of lattice paths staying below the line $x = \mu y$, one defines the generating functions

$$\begin{aligned} F_0(z) &= \sum_{n\geq 0} |L((0,0) \to (\mu n, n) \mid x \geq \mu y)| z^n, \\ F_k(z) &= \sum_{n\geq 0} |L((0,0) \to (\mu n + k, n) \mid x \geq \mu y)| z^n, \end{aligned}$$

where $F_0(0) = F_k(0) = 1$.

Combinatorial argument shows that

$$F_0(z) = 1 + zF_0(z)^{\mu+1},$$

and

$$F_k(z) = F_0(z)^{k+1}.$$

Find formulas for $|L((0,0) \to (\mu n, n) \mid x \geq \mu y)|$ and $|L((0,0) \to (\mu n + k, n) \mid x \geq \mu y)|$.