MATH 630. Combinatorics, Fall 2008

Assignment 2. Due on Wednesday, September 24, 2008

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1. Let $A(n,k)$ be the Eularian number, that is, $A(n,k)$ is the number of permutations of length n with $k-1$ descents. Prove that $A(n,k)$ satisfies the recurrence

$$A(n,k) = kA(n-1,k) + (n-k+1)A(n-1,k-1), \quad \text{for } 2 \le k \le n,$$

with boundary conditions $A(n,0) = 0$, $A(n,1) = A(n,n) = 1$ and $A(n,k) = 0$ for $k > n$.

2. Again $A(n,k)$ is the Eularian number. Prove that

$$A(n,k) = A(n, n+1-k).$$

3. Let $S \subseteq [n-1]$, and let $\alpha(S)$ denote the number of n-permutations whose descent set is contained in S.

Find the one-element set $\{i\} \subseteq [n-1]$ for which $\alpha(\{i\})$ is maximal.

4. Prove that for any permutation π, $i(\pi) = i(\pi^{-1})$, where $i(\pi)$ is the number of inversions of π.

5. A permutation π is called *even* if $i(\pi)$ is even. Similarly, π is called *odd* if $i(\pi)$ is odd. Let $n \ge 2$. Prove that the number of even (odd) permutations of length n is $n!/2$.

6.Exercises on page 49, Problem 31. For a permutation π, let $m(\pi)$ denote the number of left-to-right maxima of π and $i(\pi)$ the number of inversions of π. Compute the generating function

$$F(x,q) = \sum_{\pi \in \mathcal{S}_n} x^{m(\pi)} q^{i(\pi)}.$$

Please state your reason.

7. The order of a permutation π is the smallest positive integer k for which $\pi^k = id$. Assume that *pi* is of cycle type $(c_1, c_2, \ldots, c_n)$. What is the order of π?

8. How many permutations has length 6 whose fourth power is the identity permutation?