MATH 630–600. Enumerative Combinatorics

Assignment 4. Due on Wednesday, October 22, 2008

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1. Let $D(n)$ be the number of derangement of length n. Prove that the generating function of $D(n)$ satisfies
$$\sum_{n\geq 0} \frac{D(n)x^n}{n!} = \frac{e^{-x}}{(1-x)}.$$

2. Let $m > n$ be two positive integers. How many ballot sequences $(a_1, a_2, \ldots, a_{m+n})$ such that
   (a) $a_i \in \{+1, -1\}$,
   (b) $a_1 + a_2 + \cdots + a_i > 0$ for all $i = 1, 2, \ldots, m+n$,
   (c) $a_1+a_2+\cdots+a_{m+n} = m-n$. That is, the multiset $\{a_1, \ldots, a_{m+n}\} = \{(+1)^m, (-1)^n\}$.

3. Given a sequence $(a_n)_{n\geq 0}$, let $A(x) = \sum_{n\geq 0} a_n x^n$ be its generating function.

   Suppose that the sequence (a_n) is the binomial transform of a sequence (b_n), i.e.,
$$a_n = \sum_{i=0}^{n} \binom{n}{i} b_i \qquad \forall n \geq 0.$$
   (a) Show that $A(x) = \frac{1}{1-x} B(\frac{x}{1-x})$.
   (b) Deduce that $B(x) = \frac{1}{1+x} A(\frac{x}{1+x})$.
   (c) Deduce from (b) that
$$b_n = \sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} a_i, \qquad \forall n \geq 0.$$

   This is another proof of the binomial inversion formula.

4. A sequence of positive integers $(a_1, a_2, \dots, a_n)$ is said to be of *restricted growth* if

$$a_1 = 1, \quad a_{i+1} \leq 1 + \max\{a_1, \dots, a_i\},$$

for all $i = 1, 2, \dots, n$.

 (a) Write all restricted growth sequences for $n = 4$.

 (b) Show that the number of restricted growth sequences is the Bell number B_n.

 (c) Describe the Stirling numbers $S(n, k)$ in terms of restricted growth sequences.

5. Show by a combinatorial argument that

 (a)

$$\binom{n}{n-k}_q = q^{k(n-k)} \binom{n}{k}_{q^{-1}},$$

 where $\binom{n}{k}_{q^{-1}}$ is obtained from $\binom{n}{k}_q$ by replacing q with q^{-1}. (Hint: use lattice paths from (0.0) to $(n-k, k)$).

 (b) Deduce that $\binom{n}{k}_q$ is a symmetric polynomial of q, that is, if

$$\binom{n}{k}_q = a_0 + a_1 q + a_2 q^2 + \cdots + a_N q^N$$

 with $a_N \neq 0$, then $a_i = a_{N-i}$ for all i.

6. Vandermonde's formula for the q-binomial coefficients is

$$\binom{n+m}{p}_q = \sum_{k=0}^{p} q^{(m-k)(p-k)} \binom{m}{k}_q \binom{n}{p-k}_q.$$

 Prove this formula by a lattice path counting argument.

 (Hint: Count lattice paths from $(0, 0)$ to $(p, m+n-p)$, and consider the intersection between such paths with the line $x + y = n$.)

7. Exercise 1 on textbook, page 86.

8. Exercise 2a on textbook, page 87.

9. Exercise 7 on textbook, page 88.