MATH 630–600. Enumerative Combinatorics

Assignment 5. Due on Wednesday, November 5, 2008

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1. Let $m_1 \geq m_2 \geq \cdots \geq m_n$ be a sequence of positive integers, and $M = (m_{i,j})$ be an $n \times n$ matrix whose ijth entry is
$$m_{ij} = \frac{1}{(m_i - i + j)!}.$$
(Assume that all $m_i - i + j$ are nonnegative.) Prove that the determinant of M is
$$\det(M) = \frac{\prod_{1 \leq i < j \leq n}(m_i - m_j + j - i)}{(m_1 + n - 1)!(m_2 + n - 2)! \cdots m_n!}.$$

2. Find the rank-generating function $F(P, q)$ for the following posets.
   (a) D_N, the set of all positive integral divisors of n, where $i \leq j$ if i is a divisor of j. Assume that the prime factorization of n is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, \ldots, p_k$ are distinct primes, and α_i are positive integers.
   (b) The set Π_n of all partitions of $[n]$, ordered by refinement.
   (c) The set $L_n(q)$ that consists of all subspaces of an n-dimensional vector space $V_n(q)$ over the q-element field $\mathbb{F}_q$, ordered by inclusion.

3. If posets P and Q are graded with rank generating functions $F(P, q)$ and $F(Q, q)$, then prove
$$F(P \times Q, q) = F(P, q)F(Q, q),$$
and
$$F(P \otimes Q, q) = F(P, q^{r+1})F(Q, q).$$

4. Check the following rules of cardinal arithmetic:
$$R^{P+Q} = R^P \times R^Q,$$
$$(R^Q)^P = R^{Q \times P}.$$

5. Construct an infinite meet-semilattice P with $\hat{1}$, such that P is not a lattice.

6. (a). Prove that a finite poset of size at least $mn+1$ contains a chain of length $m+1$ or an antichain of size $n+1$.

   (b). Use (a) to show that in any finite sequence of distinct integers $a_1, a_2, \ldots, a_{n^2+1}$, there is a monotone subsequence of length at least $n+1$. Here a *monotone subsequence of length* k consists of terms $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ for some $1 \leq i_1 < i_2 < \cdots < i_k$ such that either

$$a_{i_1} < a_{i_2} < \cdots < a_{i_k} \text{ or } a_{i_1} > a_{i_2} > \cdots > a_{i_k}.$$

7. Exercises 8(a–e) on textbook, page 87. (f is optional).

8. Exercise 10 on textbook, page 89.

9. Exercise 4 on textbook, page 154.

10. Exercise 5 on textbook, page 154. Note that in part (b), assume that the Hasse disgram of P has no isolated point.

    (Could you find an answer for 5a that is different from the one given in the solution?)