MATH 630–600. Enumerative Combinatorics

Assignment 7. Due on Wednesday, December 3, 2008

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1. Let n be a square-free integer, $n = p_1 p_2 \cdots p_k$ with the p_i distinct primes. Show that the maximum number of divisors of n which do not divide one another is $\binom{k}{\lfloor k/2 \rfloor}$.

2. Fix $k \in \mathbb{N}$. For each $n \in \mathbb{Z}^+$, let
$$\sigma_k(n) = \sum_{d|n} d^k.$$
   (a) Check that $\sigma_0(n) = \nu(n)$ (the number of divisors of n) and $\sigma_1(n) = \sigma(n)$ (the sum of the divisors of n).
   (b) Let $p_1, \ldots, p_r$ be the different prime divisors of n. Show that
$$n^k = \sigma_k(n) - \sum_i \sigma_k(\frac{n}{p_i}) + \sum_{i<j} \sigma_k(\frac{n}{p_i p_j}) + \cdots + (-1)^r \sigma_k(\frac{n}{p_1 \cdots p_r}).$$
   (c) Verify this identity when $n = 12$ and $k = 2$.

3. For each $n \in \mathbb{Z}^+$, let
$$\phi(n) := \#\{i \in [n] : \gcd(i, n) = 1\}.$$
   This is the *Euler function.* For each positive divisor d of n, let
$$\Phi(d, n) := \{i \in [n] : \gcd(i, n) = d\}.$$
   (a) Show that there is a disjoint decomposition
$$[n] = \bigcup_{d|n} \Phi(d, n),$$
   and a bijection $\Phi(d, n) \cong \Phi(1, n/d)$.
   (b) Deduce that
$$n = \sum_{d|n} \phi(d).$$
   (c) Deduce that
$$\phi(n) = n \prod_{p|n} (1 - \frac{1}{p}),$$
   where p ranges over prime factors of n.

4. Given a subspace V of $\mathbb{F}_q^n$, let $\alpha(V)$ be the number of subsets of V and $\beta(V)$ the number of spanning subsets of V. Here a subset W of V is *spanning* if V equals the vector space spanned by all the vectors in W. (On the other hand, the vectors in W are not necessarily independent.)

(a) Show that

$$\alpha(V) = 2^{q^{\dim(V)}} \qquad \text{and} \qquad \alpha(V) - 1 = \sum_{U \le V} \beta(U).$$

(Note that the empty subset of V does not span a subspace.)

(b) Deduce that the number of spanning subsets of $\mathbb{F}_q^n$ is

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} (2^{q^k} - 1).$$

5. Let $S = \{s_1, s_2, \ldots, \}$ be a set of positive integers. Let $h_S(n)$ be the number of partitions of the set $[n]$ into blocks so that each block size is an element of S. Let $H_S(x)$ be the exponential generating function of the sequence $\{h_S(n)\}$. Prove that

$$H_S(x) = \exp(\sum_{i \ge 1} \frac{x^{s_i}}{s_i!}).$$

6. An involution is a permutation π such that $\pi^2 = id$. Let $i(n)$ be the number of involutions of length n. Compute the exponential generating function for the sequence $\{i(n)\}$.

7. A *threshold graph* is a simple (i.e. no loops or multiple edges) graph which may be defined inductively as follows:

(a) The empty graph is a threshold graph.

(b) If G is a threshold graph, then so is the disjoint union of G with a one-vertex graph.

(c) If G is a threshold graph, then so is the (edge) complement of G.

Let $t(n)$ be the number of threshold graphs with vertex set $[n]$, with $t(0) = 1$, and let $s(n)$ denote the number of such graphs with no isolated vertex, (so $s(0) = 1$, $s(1) = 0$). Set $T(x) = E_t(x)$ and $S(x) = E_s(x)$.

(a) List all threshold graph on [4], and compute $t(n)$, $s(n)$ for $n = 2, 3, 4$.

(b) Show that

$$T(x) = e^x S(x), \qquad \textit{and} \qquad T(x) = 2S(x) + x - 1.$$

(c) Deduce that

$$\begin{aligned} T(x) &= e^x(1-x)/(2-e^x), \\ S(x) &= (1-x)/(2-e^x). \end{aligned}$$

8. Find the unique power series $F(x)$ such that for all $n \in \mathbb{N}$, we have $[x^n]F(x)^{n+1} = 1$.

9. (Optional) A tree on $\{0, 1, \ldots, n\}$ is called *alternating* if for every vertex i all neighbors are either greater than i, or all are smaller than i. Let h_n be the number of alternating trees on $\{0, 1, \ldots, n\}$.

   (a) List all the alternating trees for h_2.

   (b) Prove that $H(z) = \sum_{n \geq 0} h_n \frac{z^n}{n!}$ satisfies the equation

   $$H(z) = e^{\frac{z}{2}(H(z)+1)}.$$

   (c) Deduce from above that

   $$h_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}.$$