

MATH 630 Enumerative Combinatorics

Solution of Assignment 1.

1. Let $f(n)$ be a sequence with the following exponential generating function

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \exp\left(x + \frac{x^2}{2}\right).$$

Evaluate the sum

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} f(i).$$

Solution. Let $g(n) = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} f(i)$, and $G(x) = \sum_{n \geq 0} g(n) x^n / n!$. Since $g(n)$ is the convolution of $f(n)$ and $a_n = (-1)^n$, $G(x) = e^{-x} F(x) = e^{x^2/2} = \sum_{n \geq 0} (x^2/2)^n / n!$. Thus $g(n) = 0$ if n is odd, and it is $n! / (2^{n/2} n!) = (n-1)!!$ if $n = 2m$.

Exercises on page 43. Solutions.

1. (a): There are 2^{10} subsets of $[10]$, among them 2^5 do not contain any odd integers. So $2^{10} - 2^5$.

1. (b): First, the number of circular permutation of $[n]$ is $(n-1)!$. In this problem, two arrangements are equivalent if one can be obtained from the other by a reflection with respect to a diameter. Hence the answer is $(n-1)!/2$. Here $n = 7$.

1. (f): $(6)_4 = 6 \times 5 \times 4 \times 3$.

1. (i): This is to count the number of permutations of the multiset $\{M, I^4, S^4, P^2\}$, where those having four S's consecutive are the permutations of the multiset $\{M, I^4, P^2, SSSS\}$, where $SSSS$ is a single symbol. Hence $\binom{11}{1,4,4,2} - \binom{8}{1,4,2,1}$.

2.(a). Prove for x positive integers. The RHS counts the number of n -subset of $[x+n+1]$. For any n -subset S of $[x+n+1]$, let $x+i+1$ be the largest integer that is not in S . Then $0 \leq i \leq n$. In this case, S contains $x+i+2, x+i+3, \dots, x+n+1$ and other i elements from $[x+i]$. There are $\binom{x+i}{i}$ many ways to choose these i elements. Hence

$$\binom{x+n+1}{n} = \sum_{i=0}^n \binom{x+i}{i}.$$

2. (b). See solution on page 52.

3. See solution on page 52.

4. For part (a), see solution. For part (b), note that $\binom{n}{k} = 0$ if n or $k < 0$, and $\binom{2n-1}{n} = \frac{1}{2} \binom{2n}{n}$ for $n > 0$. So

$$\sum_{n \geq 0} \binom{2n-1}{n} x^n = \frac{1}{2} \sum_{n > 0} \binom{2n}{n} x^n = \frac{1}{2} ((1-4x)^{-1/2} - 1).$$

5. (a). Conditioned on the form of the first step, we have the following recurrence

$$f(m, n) x^m y^n = x^1 y^0 f(m-1, n) x^{m-1} y^n + x^0 y^1 f(m, n-1) x^m y^{n-1} + x^1 y^1 f(m-1, n-1) x^{m-1} y^{n-1} \quad (1)$$

for $m \geq 1, n \geq 1$. For boundary conditions, we have

$$f(m, 0)x^m y^0 = x^1 y^0 f(m-1, n)x^{m-1} y^0, \quad m \geq 1, \quad (2)$$

$$f(0, n)x^0 y^n = x^0 y^1 f(0, n-1)x^0 y^{n-1}, \quad n \geq 1. \quad (3)$$

Sum over (1),(2),(3) for $m \geq 1, n \geq 1$. If $F(x, y) = \sum_{m \geq 0} \sum_{n \geq 0} f(m, n)x^m y^n$, then

$$F(x, y) - 1 = xF(x, y) + yF(x, y) + xyF(x, y).$$

It follows $F(x, y) = (1 - x - y - xy)^{-1}$.

Another way to do it is to use generating functions. Let $f_k(m, n)$ be the number of paths from $(0, 0)$ to (m, n) using exactly k steps, and

$$F_k(x, y) = \sum_{m \geq 0, n \geq 0} f_k(m, n)x^m y^n.$$

Clearly $F_0(x, y) = 1$, and $F_1(x, y) = x + y + xy$. Given a position (m, n) , adding one more step will reach positions $(m+1, n)$, or $(m, n+1)$, or $(m+1, n+1)$. Hence

$$F_{k+1}(x, y) = F_k(x, y)(x + y + xy).$$

So $F_k(x, y) = (x + y + xy)^k$. It follows

$$F(x, y) = \sum_{k \geq 0} F_k(x, y) = (1 - x - y - xy)^{-1}.$$

5. (b). See the solution at page 52–53. Note that by Newton’s binomial theorem,

$$(1 - x)^{-m} = \sum_{j \geq 0} \binom{-m}{j} (-x)^j = \sum_{j \geq 0} \binom{m + j - 1}{j} x^j.$$

The general method to solve this type of problem is to (1) substitute $x = s$ and $y = t/s$, and (2) use Residue theorem or partial fractions. In this particular problem, we don’t need to use those big machines! –Those methods are covered in EC, vol 2, Chapter 6, if you are interested.

10. See the solution at page 55. Problem 10 can also be done by induction.

12. Let $D_1 = T_1$, $D_i = T_i \setminus T_{i-1}$, for $1 < i \leq k$, and $D_{k+1} = S \setminus T_k$. Then (T_1, \dots, T_k) and (D_1, \dots, D_{k+1}) determine each other. And $(D_1, D_2, \dots, D_{k+1})$ is a sequence of disjoint subsets of S with $\cup_{i=1}^{k+1} D_i = S$. So it is the same as the number of functions from $|S| = n$ distinguishable “balls” to $k + 1$ distinguishable “boxes”, and the number is $(k + 1)^n$.

13. Let $b_i = a_i - (i - 1)(j - 1)$. Then there is an one-to-one correspondence between the desired sequences to (b_1, b_2, \dots, b_k) where $1 \leq b_1 < b_2 < \dots < b_k \leq n - (k - 1)(j - 1)$. There are $\binom{n - (k-1)(j-1)}{k}$ many such b sequences.

14. (a,c,e): The goal is to establish the recurrence $f(n) = f(n - 1) + f(n - 2)$, and then compare the initial values with that of Fibonacci numbers.

14. (f). The sum counts the number of ways to insert bars into the spaces between n balls such that no two bars are in the same space, and then circle one ball in each compartment. Given

such a configuration, replace each uncircled ball by a “2”, each circled ball by a “1”, and each bar by a “1”. Then we have a composition of parts “1” and “2”. For $a_1 + a_2 + \cdots + a_k = n$, there are $k - 1$ bars, k circled balls, and $n - k$ uncircled balls. So the total sum in the 1-2-composition is $k - 1 + k + 2(n - k) = 2n - 1$. Therefore we obtain a composition of $2n - 1$ into 1’s and 2’s. This correspondence is invertible. Given any composition of $2n - 1$ into 1’s and 2’s, we can recover the original bar-ball configuration: For each 1, replace it by a uncircled ball, for 2’s replace alternatively by circled balls and bars. Note that there must be an odd number of 1’s in the composition. So we will get a configuration of balls and bars in which every compartment of balls has a circled one.

The number of compositions of $2n - 1$ into 1’s and 2’s is computed in 14.(c) and it is F_{2n} .

Another way, (less cute, but may be more natural), is to look for recurrences for $h(n) = \sum a_1 a_2 \cdots a_k$. One has $h(n) = \sum_{i=1}^n i h(n - i)$, and $h(1) = 1, h(2) = 3$. (Let $h(0) = 1$). This leads to $h(n) = 3h(n - 1) - h(n - 2)$. Then it suffices to check $F_{2n} = 3F_{2n-2} - F_{2n-4}$.