MATH 689. Enumerative Combinatorics
Solution of Assignment 2.

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1. Let $A(n,k)$ be the Eularian number, that is, $A(n,k)$ is the number of permutations of length n with $k-1$ descents. Prove that $A(n,k)$ satisfies the recurrence

$$A(n,k) = kA(n-1,k) + (n-k+1)A(n-1,k-1), \quad \text{for } 2 \le k \le n,$$

with boundary conditions $A(n,0)=0$, $A(n,1)=A(n,n)=1$ and $A(n,k)=0$ for $k>n$.
Solution. For a permutation $\pi = a_1a_2\dots a_n \in \mathcal{S}_n$, assume $n = a_i$.
Case 1. If $i=n$, or $i \ne n$ and $a_{i-1} > a_{i+1}$, then removing n will yield a permutation of length $n-1$ with $k-1$ descents. There are $A(n-1,k)$ such permutations in $\mathcal{S}_{n-1}$. Conversely, for each such a permutation, one can insert n back in any of the descent position, or at the position n. There are k such choices. Therefore, $kA(n-1,k)$ permutations in $\mathcal{S}_n$ fall into this case.
Case 2. If $i \ne n$ but $a_{i-1} < a_{i+1}$, then removing n will yield a permutation of length $n-1$ with $k-2$ descents. Conversely, for each such a permutation, one can insert n into the first position, or any of the non-descent positions. There are $(n-1)-(k-2) = n-k+1$ choices. Hence there are $(n-k+1)A(n-1,k-1)$ permutations in $\mathcal{S}_n$ fall into this case.

2. Again $A(n,k)$ is the Eularian number. Prove that

$$A(n,k) = A(n, n+1-k).$$

Solution. If $a_1a_2\cdots a_n$ has $k-1$ descents, then $a_na_{n-1}\cdots a_2a_1$ has $n-1-(k-1) = n-k$ descents.

3. Let $S \subseteq [n-1]$, and let $\alpha(S)$ denote the number of n-permutations whose descent set is contained in S.

Find the one-element set $\{i\} \subseteq [n-1]$ for which $\alpha(\{i\})$ is maximal.
Solution. $\alpha(\{i\}) = \binom{n}{i}$. Hence the maximal is reached when $i = \lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$.

4. Prove that for any permutation π, $i(\pi) = i(\pi^{-1})$, where $i(\pi)$ is the number of inversions of π.
Solution. If $(\pi(i), \pi(j))$ is an inversion of π iff $i<j$ and $\pi(i) > \pi(j)$. But this is exactly same as positions $\pi(j) < \pi(i)$, and $\pi^{-1}(\pi(j)) > \pi^{-1}(\pi(i))$.

5. A permutation π is called *even* if $i(\pi)$ is even. Similarly, π is called *odd* if $i(\pi)$ is odd. Let $n \ge 2$. Prove that the number of even (odd) permutations of length n is $n!/2$.
Solution. For any $\pi \in \mathcal{S}_n$ with $\pi = a_1a_2a_3\cdots a_n$, let $f(\pi)$ be the permutation obtained from π by exchanging the first two entries, i.e., $f(\pi) = a_2a_1a_3\cdots a_n$. Then f is an involution of $\mathcal{S}_n$ which maps even permutations to odd permutations. Hence the number of even permutations is the same as the number of odd permutations, and is equal to $n!/2$.

6.Exercises on page 49, Problem 31. For a permutation π, let $m(\pi)$ denote the number of left-to-right maxima of π and $i(\pi)$ the number of inversions of π. Compute the generating function

$$F(x,q) = \sum_{\pi \in \mathcal{S}_n} x^{m(\pi)} q^{i(\pi)}.$$

Please state your reason.
Solution. Consider the inversion table $(b_1, b_2, \ldots, b_n)$ for π. A number i is a left-to-right maximal iff $b_i = 0$. Hence

$$F(x,q) = \sum_{(b_1,b_2,\ldots,b_n)} x^{\sum_i \chi(b_i=0)} q^{\sum_i b_i},$$

where $0 \leq b_i \leq n-i$, and $\chi(b_i = 0)$ is the indicator of $b_i = 0$, which is 1 if $b_i = 0$, and is 0 otherwise. Separating variables, we get

$$\begin{aligned} F(x,q) &= \left(\sum_{b_1=0}^{n-1} x^{\chi(b_1=0)} q^{b_1}\right)\left(\sum_{b_2=0}^{n-2} x^{\chi(b_2=0)} q^{b_2}\right)\cdots\left(\sum_{b_n=0}^{n-n} x^{\chi(b_n=0)} q^{b_n}\right) \\ &= \prod_{i=1}^{n}(x + q + q^2 + \cdots + q^{i-1}) \end{aligned}$$

7. The order of a permutation π is the smallest positive integer k for which $\pi^k = id$. Assume that pi is of cycle type $(c_1, c_2, \ldots, c_n)$. What is the order of π?
Solution. For a cycle C of length k, $C^t = id$ iff t is a multiple of k. Hence the order of π with cycle type $(c_1, c_2, \ldots, c_n)$ is the least common multiple of all k where $c_k \neq 0$.

8. How many permutations has length 6 whose fourth power is the identity permutation?
Solution. For a permutation π to satisfy $\pi^4 = id$, the length of each cycle in π has to be a divisor of 4, that is, 4 or 2 or 1.

1. Cycles of π have length 4 and 2: There are $6!/(4 \cdot 2) = 90$ permutations of this type.
2. Cycles of π have length $4, 1, 1$: There are $6!/(4 \cdot 1 \cdot 1 \cdot 2!) = 90$ permutations of this type.
3. Cycles of π have length $2, 2, 2$: There are $6!/(2^3 3!) = 15$ permutations of this type.
4. Cycles of π have length $2, 2, 1, 1$: There are $6!/(2^2 2! 2!) = 45$ permutations of this type.
5. Cycles of π have length $2, 1, 1, 1, 1$. There are $6!/(2 \cdot 4!) = 15$ permutations of this type.
6. Cycles of π have length $1, 1, 1, 1, 1, 1$. There is only one permutation of this type.

The total number is 256.