

MATH 630–600. Enumerative Combinatorics

Solution of Assignment 3.

1. Verify the bijective proof given on Page 40 of Enumerative Combinatorics, between the set of partitions of n into odd parts, and the set of partitions of n into distinct parts, where n is a fixed positive integer.

Proof. Let $\mathcal{A} = \{\lambda \vdash n : \lambda \text{ has odd parts only}\}$, and $\mathcal{B} = \{\lambda \vdash n : \lambda \text{ has distinct parts}\}$. The book defines a map ϕ from \mathcal{A} to \mathcal{B} as follows. Given $\lambda \in \mathcal{A}$, assume $\lambda = \langle 1^{\beta_1}, 3^{\beta_2}, \dots, (2j-1)^{\beta_j}, \dots \rangle$ where $\beta_1 + 3\beta_2 + \dots + (2j-1)\beta_j + \dots = n$. Write each β_j as a sum of distinct powers of 2 according to its binary expansion, i.e.,

$$\beta_j = 2^{\alpha_{j,1}} + 2^{\alpha_{j,2}} + \dots + 2^{\alpha_{j,k_j}},$$

where $\alpha_{j,1} < \alpha_{j,2} < \dots < \alpha_{j,k_j}$. Then $\phi(\lambda)$ is the partition whose parts are $\{(2j-1)2^{\alpha_{j,i}} : 1 \leq i \leq k_j\}$. Those parts are distinct since each of them is in the form $q2^p$ where q is odd. Two numbers $q2^p$ and $q'2^{p'}$ are equal iff $q = q'$ and $p = p'$.

To verify that ϕ is a bijection, we find ϕ^{-1} . Given $\mu \vdash n$, $\mu \in \mathcal{B}$, assume $\mu = (\mu_1, \mu_2, \dots)$, where $\mu_1 + \mu_2 + \dots = n$. Each μ_i can be written uniquely as $q_i 2^{p_i}$ where q_i is odd. Then $n = q_1 2^{p_1} + q_2 2^{p_2} + \dots$. Note that if $q_i = q_j$ for some $i \neq j$, then $p_i \neq p_j$ since $\mu_i \neq \mu_j$. Now $\phi^{-1}(\mu)$ is the partition whose parts are

$$\underbrace{q_1, q_1, \dots, q_1}_{2^{p_1} \text{ many}}, \underbrace{q_2, q_2, \dots, q_2}_{2^{p_2} \text{ many}}, \dots, \underbrace{q_j, q_j, \dots, q_j}_{2^{p_j} \text{ many}}, \dots$$

2. Let $\binom{n}{k}_q$ be the Gaussian coefficient. Prove the following identity:

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q.$$

Proof. The LHS counts the number of k -dimensional subspaces of $V_n(q)$, an n -dimensional vector space over a finite field \mathbb{F}_q of q elements. Fix an orthogonal basis $\langle x_1, x_2, \dots, x_n \rangle$ of $V_n(q)$, and let $W = \langle x_1, x_2, \dots, x_{n-1} \rangle$. Let U be a k -dimensional subspace.

Case I. U contains the 1-dimensional subspace $\langle x_n \rangle$. Then $U' = U \cap W$ is a $(k-1)$ -subspace of W . Conversely, any $(k-1)$ -subspace U' can be uniquely extended to a k -subspace of $V_n(q)$ by taking the span of U' and x_n . Hence there are $\binom{n-1}{k-1}_q$ many such k -subspaces.

Case II. U does not contain $\langle x_n \rangle$. Then the projection of U to W , denoted by $\pi(U)$, is a k -subspace of W . Conversely, any k -subspace W' of W can be raised to q^k many k -subspaces of $V_n(q)$: Take a basis w_1, \dots, w_k of W' , then $\pi^{-1}(w_i) = w_i + c_i x_n$ where c_i can be any number of \mathbb{F}_q . Therefore there are $q^k \binom{n-1}{k}_q$ many such k -subspaces.

Combining Case I and II one gets the identity.

One can also prove it by counting lattice paths with a box of $k \times (n-k)$. The LHS is the area enumerator of all such paths. Consider the last step. If the last step is $(1, 0)$, then the earlier path is one from $(0, 0)$ to $(k-1, n-k)$. There are $\binom{n-1}{k-1}_q$ many such paths, whose area-enumerator is $\binom{n-1}{k-1}_q$. If the last step is $(0, 1)$, the earlier path is one from $(0, 0)$ to $(k, n-k-1)$, the area-enumerator for the earlier segment is $\binom{n-1}{k}_q$, and the last $(0, 1)$ step contributes k more squares. Hence the total contribution for such paths is $q^k \binom{n-1}{k}_q$.

3. Let $p_k(n)$ be the number of partitions of integer n with exactly k parts. Compute the generating function

$$\sum_{n \geq 0} p_k(n)q^n.$$

Solution. Taking the conjugate, a partition with exactly k parts becomes a partition whose largest parts is k , i.e., partitions who has x_i many parts of size i , for $i = 1, 2, \dots$, where all x_i non-negative, and $x_k \geq 1$. Hence

$$\sum_{n \geq 0} p_k(n)q^n = x^k \prod_{i=1}^k \frac{1}{1 - q^i}.$$

4. Let $S(n, k)$ be the number of partitions of an n -set into k blocks, (i.e., the Stirling number of the second kind.) Prove that

$$\sum_{n \geq k} S(n, k)x^n = \frac{x^k}{(1-x)(1-2x) \cdots (1-kx)}.$$

Solution. A combinatorial proof is available as Problem 16, page 46 in the textbook. Let $F_k(x) = \sum_{n \geq k} S(n, k)x^n$. We give an inductive proof based on the recurrence

$$S(n, k) = kS(n-1, k) + S(n-1, k-1).$$

Multiply both side by x^n and sum over all $n \geq k$, we get

$$F_k(x) = kxF_k(x) + xF_{k-1}(x).$$

Hence $F_k(x) = \frac{x}{1-kx}F_{k-1}(x)$. The claim follows from the initial condition $F_0(x) = 1$ and $F_1(x) = x/(1-x)$.

Problems 20 on textbook, page 47.

See solution on page 57.

Problem 23(a, b, c, d) on page 48

For proofs, see the solution on page 58. The key point is to compare the coefficient of $q^k x^n$ in both sides of the identities, and find combinatorial interpretations.

Note that (d) can be proved from (c) by making the substitution $x \rightarrow x^2$ and $q \rightarrow qx^{-1}$. Various substitutions are commonly used in proving identities involving infinite summations or products.

Problem 42 on page 50.

Solution. Take $b_i = (a_i - i)/2 + 1$. Then b_i are non-decreasing integer sequences between 1 and $m = \lfloor (n-k)/2 + 1 \rfloor$. There are $\binom{m+k-1}{k} = \binom{\lfloor (n+k)/2 \rfloor}{k}$ many such sequences.