MATH 630–600. Enumerative Combinatorics
Solution of Assignment 4.

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1. Let $D(n)$ be the number of derangement of length n. Prove that the generating function of $D(n)$ satisfies
$$\sum_{n\geq 0}\frac{D(n)x^n}{n!}=\frac{e^{-x}}{(1-x)}.$$
Solution. All formulas on page 67 of the book can lead to this equation. For example, from formula (11), one notices that $D(n)$ is the convolution of $a_n=n!$ and $b_n=(-1)^n$. Hence passing to the exponential generating functions, one has $\sum D(n)\frac{x^n}{n!}=\sum n!\frac{x^n}{n!}\sum(-1)^n\frac{x^n}{n!}=e^{-x}/(1-x)$.

2. Let $m>n$ be two positive integers. How many ballot sequences $(a_1,a_2,\ldots,a_{m+n})$ such that
   (a) $a_i\in\{+1,-1\}$,
   (b) $a_1+a_2+\cdots+a_i>0$ for all $i=1,2,\ldots,m+n$,
   (c) $a_1+a_2+\cdots+a_{m+n}=m-n$. That is, the multiset $\{a_1,\ldots,a_{m+n}\}=\{(+1)^m,(-1)^n\}$.

   Solution. Any such ballot sequence must $a_1=1$. Hence we can count the number of subdiagonal paths from $A=(1,0)$ to $P=(m,n)$. The total number of lattice paths from A to P is $\binom{m+n-1}{n}$. For those that touches the line $y=x$, by reflecting with respect to the last intersection of the path and the line $y=x$, are in one-to-one correspondence with the lattice paths from A to $Q=(n,m)$. There are $\binom{m+n-1}{m}$ many such paths. So the number that stay below $y=x$ is
$$\binom{m+n-1}{n}-\binom{m+n-1}{m}=\frac{m-n}{m+n}\binom{m+n}{n}.$$

3. Given a sequence $(a_n)_{n\geq 0}$, let $A(x)=\sum_{n\geq 0}a_nx^n$ be its generating function.

   Suppose that the sequence (a_n) is the binomial transform of a sequence (b_n), i.e.,
$$a_n=\sum_{i=0}^{n}\binom{n}{i}b_i\qquad\forall n\geq 0.$$
   (a) Show that $A(x)=\frac{1}{1-x}B(\frac{x}{1-x})$.
   (b) Deduce that $B(x)=\frac{1}{1+x}A(\frac{x}{1+x})$.
   (c) Deduce from (b) that
$$b_n=\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}a_i,\qquad\forall n\geq 0.$$
   This is another proof of the binomial inversion formula.

   Solution.

(a) One checks that $(1-x)A(x) = \sum_n (a_n - a_{n-1})x^n$, where

$$a_n - a_{n-1} = \sum_{i=0}^{n}(\binom{n}{i} - \binom{n-1}{i})b_i = \sum_{i=0}^{n-1}\binom{n-1}{i-1}b_i.$$

On the other hand,

$$B(\frac{x}{1-x}) = \sum_i b_i(\frac{x}{1-x})^i = \sum_i b_i x^i(\sum_j \binom{i+j-1}{j}x^j).$$

The coefficient of x^n in $B(\frac{x}{1-x})$ is obtained by summing over all indices i, j with $i+j=n$, which gives

$$\sum_{i+j=n} b_i\binom{i+j-1}{j} = \sum_{i=0}^{n-1}\binom{n-1}{i-1}b_i.$$

(b) Let $y = x/(1-x)$, then $x = y/(y+1)$. Substitute into (a).

(c) We compute

$$\begin{aligned}
\frac{1}{1+x}A(\frac{x}{1-x}) &= (\sum_i (-1)^i x^i)[\sum_j a_j x^j(\sum_k (-1)^k\binom{j+k-1}{k}x^k)] \\
&= (\sum_i (-1)^i x^i)(\sum_{j+k=m}(-1)^k a_j\binom{j+k-1}{k}x^{j+k}) \\
&= (\sum_i (-1)^i x^i)(\sum_m (-1)^{m-j}a_j\binom{m-1}{k}x^m)
\end{aligned}$$

Hence the coefficient of x^n in the above formula is

$$\sum_{m=0}^{n}(-1)^{n-m}(\sum_{j=0}^{m}(-1)^{m-j}a_j\binom{m-1}{j-1}) = \sum_{m=0}^{n}\sum_{j=0}^{m}(-1)^{n-j}a_j\binom{m-1}{j-1} = \sum_{j=0}^{n}(-1)^{n-j}a_j\binom{n}{j}.$$

The last equation uses the identity $\sum_{m=j}^{n}\binom{m-1}{j-1} = \binom{n}{j}$.

4. A sequence of positive integers $(a_1, a_2, \ldots, a_n)$ is said to be of *restricted growth* if

$$a_1 = 1, \quad a_{i+1} \le 1 + \max\{a_1, \ldots, a_i\},$$

for all $i = 1, 2, \ldots, n$.

(a) Write all restricted growth sequences for $n = 4$.

(b) Show that the number of restricted growth sequences is the Bell number B_n.

(c) Describe the Stirling numbers $S(n,k)$ in terms of restricted growth sequences.

Solution. For (b), define a bijection from the restricted growth sequences to the set of partitions of $[n]$ as follows: given a restricted growth sequence $a_1, a_2, \ldots, , a_n$, let $B_i = \{j : a_j = i\}$. Then $\{B_1, B_2, \cdots B_k\}$ is a partition of $[n]$, where k is the maximal value of $a_1, a_2, \ldots, a_n$. Conversely, given a partition π of $[n]$, list the blocks of π by the numerical value of the minimal element in the blocks, and assume the blocks are $B_1, B_2, \ldots, B_k$. Then let $a_i = j$ if the integer

i is in the block B_j. Clearly $a_1 = 1$. In general, if $i+1$ is the minimal element of a new block, then $a_{i+1} = 1 + \max\{a_1, \ldots, a_i\}$, otherwise a_{i+1} is the same as one of $a_1, \ldots, a_i$. This proves that the sequence $a_1, \ldots, a_n$ has restricted growth.

(c) $S(n,k)$ is the number of restricted growth sequences whose maximal term is k.

5. Show by a combinatorial argument that

(a)

$$\binom{n}{n-k}_q = q^{k(n-k)} \binom{n}{k}_{q^{-1}},$$

where $\binom{n}{k}_{q^{-1}}$ is obtained from $\binom{n}{k}_q$ by replacing q with q^{-1}.

(Hint: use lattice paths from $(0,0)$ to $(n-k,k)$).

Solution. For each lattice path P from $(0,0)$ to $(n-k,k)$, let $area(P)$ be the area between P, y-axis, and the line $y=k$, let $area^c(P)$ be the area between P, x-axis, and the line $x = n-k$. Then

$$area(P) + area^c(P) = k(n-k).$$

Hence

$$\binom{n}{n-k}_q = \sum_P q^{area(P)} = \sum_P q^{k(n-k)-area^c(P)} = q^{k(n-k)} \sum_P (q^{-1})^{area^c(P)} = q^{k(n-k)} \binom{n}{k}_{q^{-1}}.$$

(b) Deduce that $\binom{n}{k}_q$ is a symmetric polynomial of q, that is, if

$$\binom{n}{k}_q = a_0 + a_1 q + a_2 q^2 + \cdots + a_N q^N$$

with $a_N \neq 0$, then $a_i = a_{N-i}$ for all i.

Solution. We know that $\binom{n}{k}_q = \binom{n}{n-k}_q$. Hence part (a) implies that

$$\binom{n}{k}_q = q^{k(n-k)} \binom{n}{k}_{q^{-1}},$$

Substituting the formula $\binom{n}{k}_q = a_0 + a_1 q + a_2 q^2 + \cdots + a_N q^N$, and comparing the coefficients, we have $N = (n-k)k$ and $a_i = a_{N-i}$.

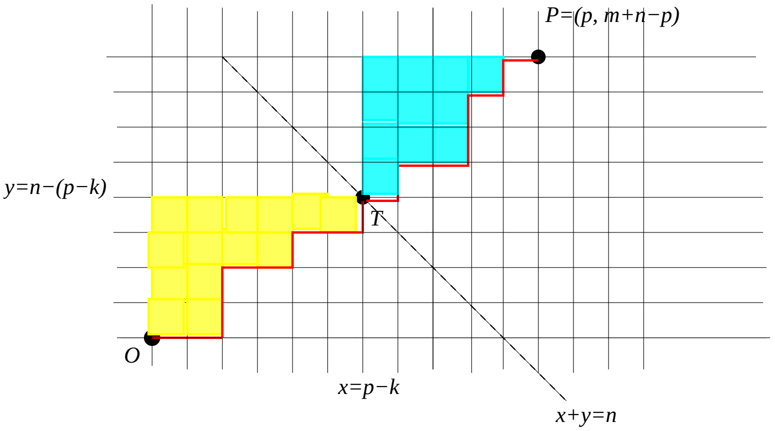
6. Vandermonde's formula for the q-binomial coefficients is

$$\binom{n+m}{p}_q = \sum_{k=0}^{p} q^{(m-k)(p-k)} \binom{m}{k}_q \binom{n}{p-k}_q.$$

Prove this formula by a lattice path counting argument.

(Hint: Count lattice paths from $(0,0)$ to $(p, m+n-p)$, and consider the intersection between such paths with the line $x+y=n$.)

Solution.



We count lattice paths from $O(0,0)$ to $P(p, m+n-p)$ by its area. Assume this path intersects the line $x+y=n$ at $T=(p-k, n-(p-k)$. Then the path P consists of two part, P_1 from O to T, and P_2 from T to P, where $area(P) = area(P_1) + area(P_2) + (p-k)(m-k)$. Hence

$$\begin{aligned}\binom{n+m}{p}_q = \sum_P q^{area(P)} &= \sum_{k=0}^{p} q^{(p-k)(m-k)} (\sum_{P_1} q^{area(P_1)})(\sum_{P_2} q^{area(P-2)}) \\ &= \sum_{k=0}^{p} q^{(m-k)(p-k)} \binom{m}{k}_q \binom{n}{p-k}_q .\end{aligned}$$

4. Exercise 1, 2a and 7 on textbook.

   Solution. **1, 2a** See solution in page 91.

   7. List all the k-subsets of $[n]$ as $C_1, C_2, \ldots, C_{\binom{n}{k}}$, and let A_i be the set of permutations which contain a cycle of length k built on C_i. Then $f_k(n) = |\overline{A_1 \cup \cdots \cup A_{\binom{n}{k}}}|$. Note that

$$|A_{i_1} \cap A_{i_2} \cdots \cap A_{i_r}| = ((k-1)!)^r (n-rk)!$$

   if $C_{i_1}, \ldots, C_{i_r}$ are disjoint. Otherwise it is 0. Hence

$$f_k(n) = \sum_{r \geq 0} (-1)^r \binom{n}{k, \ldots, k, n-rk} \frac{1}{r!} ((k-1)!)^r (n-rk)! = \sum_{r=0}^{n/k} (-1)^r \frac{n!}{i! k^i}.$$

   It follows that

$$\lim_{n \to \infty} \frac{f_k(n)}{n!} = \sum_{r \geq 0} (-1)^r \frac{1}{i! k^i} = e^{-1/k}.$$