MATH 630–600. Enumerative Combinatorics
Solution of Assignment 5.

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1. Let $m_1 \geq m_2 \geq \cdots \geq m_n$ be a sequence of positive integers, and $M = (m_{i,j})$ be an $n \times n$ matrix whose ijth entry is
$$m_{ij} = \frac{1}{(m_i - i + j)!}.$$
(Assume that all $m_i - i + j$ are nonnegative.) Prove that the determinant of M is
$$\det(M) = \frac{\prod_{1 \leq i < j \leq n}(m_i - m_j + j - i)}{(m_1 + n - 1)!(m_2 + n - 2)! \cdots m_n!}.$$
Solution. Take the factor $\frac{1}{(m_i+n-i)!}$ from the i-th row, we have a matrix N whose i, j-entry is $(y_i)_{n-j}$, where $y_i = m_i + n - i$. For the matrix N, proceed like the Vandermonde matrix: If $y_i = y_j$, then the determinant is 0. Hence the determinant has a factor of $y_i - y_j$, for each pair $i\ neqj$. Comparing the degree, one has
$$\det(N) = C \prod_{1 \leq i < j \leq n}(y_i - y_j),$$
for some constant C. Now comparing the coefficient of $y_1^{n-1} y_2^{n-2} \cdots y_n$, one gets $C = 1$.

2. Find the rank-generating function $F(P, q)$ for the following posets.

   (a) D_N, the set of all positive integral divisors of n, where $i \leq j$ if i is a divisor of j. Assume that the prime factorization of n is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, \ldots, p_k$ are distinct primes, and α_i are positive integers.

   Solution. Elements of rank n are those that can be factored as $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where each a_i is an integer in $[0, \alpha_i$, and $a_1 + a_2 + \cdots + a_k = n$. Hence $F(D_N, q)$ is the generating functions of the number of solutions to the above equation, which is
$$F(D_N, q) = \prod_{i=1}^{k}(1 + q + q^2 + \cdots + q^{\alpha_i}).$$

   (b) The set Π_n of all partitions of $[n]$, ordered by refinement.

   Solution. A partition is of rank k iff it has $n - k$ blocks. There are $S(n, n - k)$ many such partitions. Hence
$$F(\Pi_n, q) = \sum_{k=0}^{n-1} S(n, n - k) q^k.$$

   (c) The set $L_n(q)$ that consists of all subspaces of an n-dimensional vector space $V_n(q)$ over the q-element field $\mathbb{F}_q$, ordered by inclusion.

   Solution.
$$F(L_n(q), q) = \sum_{k=0}^{n} \binom{n}{k}_q q^k.$$

Note that it is not a q-analog of binomial theory, since one can not simplify the right-hand side. The q-binomial theorem is

$$(1+xq)(1+xq^2)\cdots(1+xq^n) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k+1}{2}} x^k.$$

3. If posets P and Q are graded with rank generating functions $F(P,q)$ and $F(Q,q)$, then prove

$$F(P \times Q, q) = F(P,q)F(Q,q),$$

and

$$F(P \otimes Q, q) = F(P, q^{r+1})F(Q,q).$$

Solution. Assume P is of rank n, and there are P_i many elements of rank i. Assume Q is of rank m, and there are Q_j many elements of rank j.

First it is needed to show that both posets are graded. It can be done by analyzing the structure of cover relation and maximal chains, (as we did in class). You should get that the rank of $P \times Q$ is $n+m$, and the rank of $P \otimes Q$ is $nm+m+n$.

(1) In the poset $P \times Q$, an element (x,y) is of rank t iff $\rho_P(x) + \rho_Q(y) = t$. There are $\sum_{i+j=t} P_i Q_j$ many such elements. Hence $F(P \times Q, q) = F(P,q)F(Q,q)$.

(2) In the poset $P \otimes Q$ is graded. An element (x,y) is of rank $\rho_P(x)(m+1) + \rho_Q(y)$. Hence

$$\begin{aligned} F(P \otimes Q, q) &= \sum_{x \in P, y \in Q} q^{\rho_P(x)(m+1)+\rho_Q(y)} \\ &= \sum_{x \in P} (q^{m+1})^{\rho_P(x)} \sum_{y \in Q} q^{\rho_Q(y)} \\ &= F(P, q^{r+1})F(Q,q). \end{aligned}$$

4. Check the following rules of cardinal arithmetic:

$$R^{P+Q} = R^P \times R^Q,$$

$$(R^Q)^P = R^{Q \times P}.$$

Proof. Prove by giving order-preserving bijections. For the first one, for any order-preserving map f from $P+Q$ to R, let $\theta(f) = (f|_P, f|_Q)$. For the second one, given any $f \in (R^Q)^P$, i.e., for any $p \in P$, $f(p)$ is an order-preserving map from Q to R, let $\theta(f)$ be the map from $Q \times P$ to R, whose action on $Q \times P$ is: $\theta(f) : (q,p) \to f(p)(q)$. I leave it to you to check that the above defined maps are order-preserving bijections.

5. Construct an infinite meet-semilattice P with $\hat{1}$, such that P is not a lattice.

Solution. Here is one example. Let $P = \{0, -1, -2, \cdots\} \cup \{\alpha, \beta, \hat{0}\}$ where the order is:
(1) $-n < -m$ if $n > m$ for any two natural integers n, m.
(2) $\alpha < -n$, $\beta < -n$ for all natural integers n. α and β are incomparable.
(3) $\hat{0}$ is the minimal element, which is less than anything else.
This is a meet-semilattice, where $\hat{1} = 0$. But $\alpha \vee \beta$ does not exist.

6. (a). Prove that a finite poset of size at least $mn+1$ contains a chain of length $m+1$ or an antichain of size $n+1$.

   Solution. Assume that the largest size of antichains is n. Then by Dilworth Theorem, the poset can be partitioned into n disjoint chains. Since $(mn+1)/n > m$, by pigeonhole principle, there is a chain containing at least $m+1$ many elements.

   (b). Use (a) to show that in any finite sequence of distinct integers $a_1, a_2, \ldots, a_{n^2+1}$, there is a monotone subsequence of length at least $n+1$. Here a *monotone subsequence of length* k consists of terms $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ for some $1 \le i_1 < i_2 < \cdots < i_k$ such that either

   $$a_{i_1} < a_{i_2} < \cdots < a_{i_k} \text{ or } a_{i_1} > a_{i_2} > \cdots > a_{i_k}.$$

   Solution. Construct a poset P on the elements $a_1, a_2, ..., a_{n^2+1}$ by letting $a_i \le_P a_j$ iff $i < j$ and $a_i < a_j$. Then a chain in P is an increasing subsequence, and an anti-chain is a decreasing subsequence. The claim follows from Part (a) with the case $m = n$.

7. Exercises 8(a–e) on textbook, page 87. (f is optional).

8. Exercise 10 on textbook, page 89.

   Solution. See the solutions on page 93–94.

   Note that for **8a**, the solution is similar to Problem 4 above, except that you need to define sets $A_i = \{\pi : \pi(i) = i+1\}$ for $1 \le i \le n-1$ and $A_n = \{\pi : \pi(n) = 1\}$. Also, to write PIE in a coherent summation, you need to set $(-1)! = 1$.

   8e,8f: Not very hard. But you do need to carry out some analysis. For **f**, the analysis would be finer and more complicated.

   10. First you need to transform the problem into a counting one. A typical technique to deal with "counting equivalence classes" is to find a structure such that in each equivalence class, there are a fixed number of such structures.

9. Exercise 4 on textbook, page 154.

10. Exercise 5 on textbook, page 154. Note that in part (b), assume that the Hasse diagram of P has no isolated point.

    (Could you find an answer for 5a that is different from the one given in the solution?)

    See Solution on the book for the last two problems.