MATH 630–600. Enumerative Combinatorics
Solution of Assignment 6.

---

1. Of the 16 four-element posets, (given in page 98), exactly one of them can not be built up from the poset **1** using the operations of disjoint union and ordinal sum. Find out that poset, and explain your reason.
   Solution. The poset Q given in page 101, top, right. It is connected, so not a direct sum of smaller poset. An exhausting check can verify that it can not be expressed as as ordinal sum.

2. Prove the following two lattice identities are equivalent.

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \tag{1}$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \tag{2}$$

   Solution. Assume identity (1), we prove (2).

$$\begin{aligned} \text{RHS of (2)} &= (x \wedge y) \vee (x \wedge z) \\ &= [(x \wedge y) \vee x] \wedge [(x \wedge y) \vee z] \\ &= x \wedge [(x \wedge y) \vee z] \\ &= x \wedge [(x \vee z) \wedge (y \vee z) \\ &= [x \wedge (x \vee z)] \wedge (y \vee z) \\ &= x \wedge (y \vee z). \end{aligned}$$

   The other direction is proved similarly.

3. Let V be a vector space, and $L(V)$ be the lattice of subspaces of V. Not using the dimension, prove directly that $L(V)$ is modular. That is, for any three subspaces X, Y, Z of V with $X \subseteq Z$,

$$X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z).$$

   Proof. LHS $\leq$ RHS is true in any lattice. So only need to show $LHS \geq RHS$. Given a vector $\vec{z}$ in the RHS, then $\vec{z} \in X \vee Y$ and $\vec{z} \in Z$. Hence $\vec{z} = \vec{x} + \vec{y}$ where $\vec{x} \in X$, $\vec{y} \in Y$. But $\vec{y} = \vec{z} - \vec{x} \in Z \vee X = Z$, so $\vec{y} \in Y \wedge Z$. It follows that $\vec{z} = \vec{x} + \vec{y} \in X \vee (Y \wedge Z)$.

4. Let P be a finite partially ordered set, and let r be the largest size of a chain. Then P can be partitioned into r but no fewer antichains.

   Proof. If there is a chain of size r, then since no two elements in this chain can belong to the same antichain, P cannot be partitioned into fewer than r antichains.

   Next we show that P can be partitioned into r antichains. Let $P_1 = P$ and let A_1 be the set of minimal elements of P_1. Delete the elements of A_1 from P_1 to get P_2, and let A_2 be the set of minimal elements of P_2. Note that for each element a_2 of A_2 there is an element a_1 of A_1 such that $a_1 < a_2$. Delete the elements in A_2 from P_2 to get P_3, and let A_3 be the set of minimal elements of P_3. Continue like this until the first integer m such that $A_m \neq \emptyset$, and $A_{m+1} = \emptyset$. Then $A_1, A_2, \ldots, A_m$ is a partition of P into antichains. Moreover, there is a chain

$$a_1 < a_2 < \cdots < a_m,$$

where $a_i \in A_i$, Since r is the largest size of a chain, $r \geq m$. Combining with the proof of the first part, we have $r = m$.

5. Let L be a finite lattice. Assume in L, if two elements x and y both cover $x \wedge y$, then $x \vee y$ covers both x and y. Prove that L is graded.
   Proof. This is part of Proposition 3.3.2 in page 103.

   Suppose L is not graded, and let $[u, v]$ be an interval of L of minimal length that is not graded. Then there are elements x_1, x_2 of $[u, v]$ that cover u and such that all maximal chains of each interval $[x_i, v]$ have the same length ℓ_i, where $\ell_1 \neq \ell_2$. By the condition, there are saturated chains in $[x_i, v]$ of the form $x_i < x_1 \vee x_2 < y_1 < \cdots < y_k = v$, contradicting $\ell_1 \neq \ell_2$, Hence L is graded.

6. Let B_n be the Boolean algebra of rank nm where $n = 2k+1$. Assume that $\mathcal{A}$ is an anti-chain of size $\binom{n}{k}$.

   Let $\mathcal{A}_m$ be the sub-family of $\mathcal{A}$ which contains all subsets in $\mathcal{A}$ of size m. From Sperner Theorem, we know that $\mathcal{A}_m \neq \emptyset$ iff $m = k$ or $k+1$.

   Prove that $\mathcal{A}_k = \emptyset$, or $\mathcal{A}_{k+1} = \emptyset$.

   Proof. Assume $\mathcal{A}_k \neq \emptyset$. As in class, let C be the family of subset T s.t. $|T| = k+1$, and T contains a subset of $\mathcal{A}_k$. From the proof, we know that for $|\mathcal{A}| = \binom{n}{k}$, we must have

   $$|\mathcal{A}_k|(n-k) = |C|(k+1),$$

   which means for any $S \in \mathcal{A}_k$ and $i \notin S$, $S \cup \{i\} \in C$; and for any $T \in C$, $j \in T$, $T \setminus \{j\} \in \mathcal{A}_k$. But then from a k-subset of $[n]$, we can obtain any other k-subset of $[n]$ by consecutively adding an element, then removing another element. This proves that all k-element subsets are in $\mathcal{A}_k$, which implies $\mathcal{A}_{k+1} = \emptyset$.

7. Textbook, Exercise 27a on page 158.

   Solution. Elements of $\mathbf{m} \times \mathbf{n}$ can be viewed as lattice points in the grid $\{(x, y) : 0 \leq x < m, 0 \leq y < n\}$ where the rank of (x, y) is $x + y$. Hence

   $$\prod_{x \in P} \frac{1 - q^{\rho(x)+2}}{1 - q^{\rho(x)+1}} = \prod_{0 \leq x < m, 0 \leq y < n} \frac{1 - q^{x+y+2}}{1 - q^{x+y+1}} = \binom{m+n}{m}_q.$$

   On the other hand, an order ideal of P is a subset S of lattice points in the above grid, in which if $(x, y) \in S$, then any (x', y') is in S, provided that $0 \leq x' \leq x$ and $0 \leq y' \leq y$. The rank of an order ideal equals the number of lattice points it contains. Now for each lattice point x, let Q_x be the unique unit-square that having x as the upper-right vertex. Then the union $\cup_{x \in S} Q_x$ for an order ideal S is an area in the rectangle $\{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq m-1, -1 \leq y \leq n-1\}$ that is bounded by a lattice path from $(-1, n-1)$ to $(m-1, -1)$. Hence the rank generating function of $L = J(P)$ is just the area-enumerator of lattice paths within a rectangle of size $m \times n$, which is $\binom{m+n}{m}_q$.

8. Textbook, Exercise 31 on page 160.

   The proof is outlined in page 183. Here is some detail. Let $G(x) = \sum_{y \geq x} g(y)$, then

$$\begin{aligned}
\sum_{\hat{0} \leq t \leq t} \mu(\hat{0}, x) G(t) &= \sum_{\hat{0} \leq t \leq x} \mu(\hat{0}, t) \sum_{s \geq t} g(s) \\
&= \sum_{s} g(s) \Big(\sum_{t \leq s, t \leq x} \mu(\hat{0}, t) \Big) \\
&= \sum_{s} g(s) \Big(\sum_{t \leq s \wedge x} \mu(\hat{0}, t) \Big).
\end{aligned}$$

   But

$$\sum_{t \leq s \wedge x} \mu(\hat{0}, t) = \sum_{t \in [\hat{o}, s \wedge x]} \mu(\hat{0}, t) \zeta(t, s \wedge x) = \delta(\hat{0}, s \wedge x) = \begin{cases} 1, & \text{if } s \wedge x = \hat{0}, \\ 0, & \text{otherwise.} \end{cases}$$

   So

$$\sum_{\hat{0} \leq t \leq t} \mu(\hat{0}, x) G(t) = \sum_{s : s \wedge x = \hat{0}} g(s) = f(x).$$

9. Textbook, Exercise 44 on page 162.

   See solution on page 187.

10. Textbook. Exercise 45 on page 162.

    For part (a), see solution on page 187.

    Part (b). In (62), (page 162), substituting $q \to \zeta$, $z \to -z$, and $n \to rn$, we get

$$\prod_{k=0}^{rn-1} (y - z\zeta^k) = \sum_{k=0}^{rn} (-1)^k \binom{\mathbf{rn}}{\mathbf{k}} \zeta^{\binom{k}{2}} y^{rn-k} z^k. \tag{3}$$

    Since ζ is a primitive rth root of unity, the roots of $x^r - 1 = 0$ are $1, \zeta, \zeta^2, \ldots, \zeta^{r-1}$, and $(y - z)(y - z\zeta) \cdots (y - z\zeta^{r-1}) = y^r - z^r$. When k ranges over $[0, rn-1]$, ζ^k repeats $1, \zeta, \zeta^2, \ldots, \zeta^{r-1}$ n times. Hence the left hand side of (3) equals

$$(y^r - z^r)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} y^{r(n-j)} z^{rj}. \tag{4}$$

    Comparing the right hand side of (3) and (4). They are equal as polynomials of y and z, so all the coefficients are equal. It gives $\binom{\mathbf{rn}}{\mathbf{k}} = 0$ if k is not divisible by r. When $k = rj$, $(-1)^k \binom{\mathbf{rn}}{\mathbf{k}} \zeta^{\binom{k}{2}} = (-1)^j \binom{n}{j}$, i.e.,

$$\binom{\mathbf{rn}}{\mathbf{k}} = (-1)^{j - rj} \zeta^{-\binom{rj}{2}} \binom{n}{j}.$$

Finally we need to discuss the parity of r and j. If j is even, or j and r are both odd, then both $(j - jr)/2$ and $j(rj - 1)/2$ are integers, hence $(-1)^{j-jr} = 1$ and $\zeta^{-\binom{rj}{2}} = (\zeta^r)^{-j(rj-1)/2} = 1$. If j is odd and r is even, $(-1)^{j(1-r)} = -1$, and $\zeta^{r/2} = -1$ which gives $\zeta^{-\binom{rj}{2}} = (-1)^{j(rj-1)} = -1$. In this case $(-1)^{j-rj}\zeta^{-\binom{rj}{2}} = (-1)(-1) = 1$. So we always have

$$\binom{\mathbf{rn}}{\mathbf{jr}} = \binom{n}{j}.$$