MATH 630–600. Enumerative Combinatorics
Solution of Assignment 7.

---

1. Let n be a square-free integer, $n = p_1 p_2 \cdots p_k$ with the p_i distinct primes. Show that the maximum number of divisors of n which do not divide one another is $\binom{k}{\lfloor k/2 \rfloor}$.

   Solution. Identify the divisor $m = p_{i_1} p_{i_2} \cdots p_{i_r}$ with the subset $\{i_1, i_2, \dots, i_r\}$ of $[n]$, then the poset of divisors of n, ordered by division, is isomorphic to the Boolean lattice B_n. By Sperner's Theorem, the size of a maximal antichain is $\binom{k}{\lfloor k/2 \rfloor}$.

2. Fix $k \in \mathbb{N}$. For each $n \in \mathbb{Z}^+$, let

$$\sigma_k(n) = \sum_{d|n} d^k.$$

   (a) Check that $\sigma_0(n) = \nu(n)$ (the number of divisors of n) and $\sigma_1(n) = \sigma(n)$ (the sum of the divisors of n).

   (b) Let $p_1, \dots, p_r$ be the different prime divisors of n. Show that

$$n^k = \sigma_k(n) - \sum_i \sigma_k(\frac{n}{p_i}) + \sum_{i<j} \sigma_k(\frac{n}{p_i p_j}) + \cdots + (-1)^r \sigma_k(\frac{n}{p_1 \cdots p_r}).$$

   (c) Verify this identity when $n = 12$ and $k = 2$.

   Solution. It is the classical Mobius inversion formula in number theory. We have

$$n^k = \sum_{d|n} \phi(d) \sigma_k(\frac{n}{d}), \qquad (1)$$

   where $\phi(d)$ is the Mobius function in number theory. Note that $\mu(d) = 0$ if d has a square factor,and $\mu(d) = r$ if d is the product of r different primes. Then equation (1) can be simplified by summing over those d with distinct prime factors.

   Part (c) is easy to check once you write down the formulas.

3. For each $n \in \mathbb{Z}^+$, let

$$\phi(n) := \#\{i \in [n] : \gcd(i, n) = 1\}.$$

   This is the *Euler function*. For each positive divisor d of n, let

$$\Phi(d, n) := \{i \in [n] : \gcd(i, n) = d\}.$$

   (a) Show that there is a disjoint decomposition

$$[n] = \bigcup_{d|n} \Phi(d, n),$$

   and a bijection $\Phi(d, n) \cong \Phi(1, n/d)$.

   Solution. The decomposition is obtained by classifying elements in $[n]$ by its gcd with n. For the bijection, note that $gcd(i, n) = d$ iff $i = dk$ and $gcd(k, n/d) = 1$.

(b) Deduce that

$$n = \sum_{d|n} \phi(d).$$

Solution. By the bijection in (a), $|\Phi(d,n)| = |\Phi(1, n/d)| = \phi(n/d)$. Hence

$$n = \sum_{d|n} |\Phi(d,n)| = \sum_{d|n} \phi(n/d) = \sum_{d|n} \phi(d).$$

The last equation is obtained by a change of index.

(c) Deduce that

$$\phi(n) = n \prod_{p|n} (1 - \frac{1}{p}),$$

where p ranges over prime factors of n.

Solution. By Mobius inversion formula,

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = n[1 - \sum_i \frac{1}{p_i} + \sum_{i<j} \frac{1}{p_i p_j} - \cdots + ((-1)^r \frac{1}{p_1 \cdots p_r},$$

where $p_1, \ldots, p_r$ from the set of prime factors of n, (the argument is similar to part (b) of Problem 2.) The last formula is just $n \prod_{p|n} (1 - \frac{1}{p})$.

4. Given a subspace V of $\mathbb{F}_q^n$, let $\alpha(V)$ be the number of subsets of V and $\beta(V)$ the number of spanning subsets of V. Here a subset W of V is *spanning* if V equals the vector space spanned by all the vectors in W. (On the other hand, the vectors in W are not necessarily independent.)

(a) Show that

$$\alpha(V) = 2^{q^{\dim(V)}} \qquad \text{and} \qquad \alpha(V) - 1 = \sum_{U \leq V} \beta(U).$$

(Note that the empty subset of V does not span a subspace.)

Solution. The first equation follows from the fact that there are $q^{\dim(V)}$ many vectors in V. The second one is true since all the non-empty subset of V span a subspace U of V.

(b) Deduce that the number of spanning subsets of $\mathbb{F}_q^n$ is

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} (2^{q^k} - 1).$$

Solution. By Mobius inversion formula,

$$\beta(V) = \sum_{U \in L_n(V)} \mu(U,V)(\alpha(U) - 1).$$

Now, if U is of dimension k, then $\alpha(U) = 2^{q^k}$, and $\mu(U,V) = \mu_{n-k}$ in $L_n(V)$, which is $(-1)^{n-k} q^{\binom{n-k}{2}}$.

5. Let $S = \{s_1, s_2, \ldots, \}$ be a set of positive integers. Let $h_S(n)$ be the number of partitions of the set $[n]$ into blocks so that each block size is an element of S. Let $H_S(x)$ be the exponential generating function of the sequence $\{h_S(n)\}$. Prove that

$$H_S(x) = \exp(\sum_{i\geq 1} \frac{x^{s_i}}{s_i!}).$$

Solution. This is a direct application of the exponential formula. Let $f(n)$ be the number of such partitions with exactly one block. Then $f(n) = 1$ iff $n \in S$. Hence $E_f(x) = \sum_{i\geq 1} x^{s_i}/s_i!$, and $H_S(x) = E_h(x) = \exp(E_f(x))$.

6. An involution is a permutation π such that $\pi^2 = id$. Let $i(n)$ be the number of involutions of length n. Compute the exponential generating function for the sequence $\{i(n)\}$.

Solution. An involution is a permutation with cycles of length 1 and 2 only. Using the Exponential formula, permutation version, we have the E.G.F is $\exp(x + \frac{x^2}{2})$.

7. A *threshold graph* is a simple (i.e. no loops or multiple edges) graph which may be defined inductively as follows:

   (a) The empty graph is a threshold graph.

   (b) If G is a threshold graph, then so is the disjoint union of G with a one-vertex graph.

   (c) If G is a threshold graph, then so is the (edge) complement of G.

   Let $t(n)$ be the number of threshold graphs with vertex set $[n]$, with $t(0) = 1$, and let $s(n)$ denote the number of such graphs with no isolated vertex, (so $s(0) = 1$, $s(1) = 0$). Set $T(x) = E_t(x)$ and $S(x) = E_s(x)$.

   (a) List all threshold graph on [4], and compute $t(n)$, $s(n)$ for $n = 2, 3, 4$.

   (b) Show that

   $$T(x) = e^x S(x), \qquad \textit{and} \qquad T(x) = 2S(x) + x - 1.$$

   Solution. To obtain a threshold graph G on $[n]$, choose a subset I of $[n]$ to be the set of isolated vertices of G, and choose a threshold graph without isolated vertices on $[n] - I$. This implies that $T(x) = e^x S(x)$.

   A threshold graph G with $n \geq 2$ vertices has no isolated vertices if and only if the complement $\bar{G}$ has isolated vertices. Hence $t(n) = 2s(n)$ with $n \geq 2$. Since $s() = t(0) = 1$, $t(1) = 1$ and $s(1) = 0$, it follows that $T(x) = 2S(x) + x - 1$.

   (c) Deduce that

   $$\begin{aligned} T(x) &= e^x(1-x)/(2-e^x), \\ S(x) &= (1-x)/(2-e^x). \end{aligned}$$

   Solution. This is obtained by solving $S(x)$ and $T(X)$ from the two equations in part (b).

8. Find the unique power series $F(x)$ such that for all $n \in \mathbb{N}$, we have $[x^n]F(x)^{n+1} = 1$.

   Solution. Let y be a formal power series satisfying $y = xF(y)$. By Lagrange inversion formula with $k = 1$,
$$n[x^n]y = [x^{n-1}]F(y)^n = 1.$$
   So $y = \sum_{n\geq 1} x^n/n = -\log(1-x)$. Hence $y^{<-1>} = 1-e^{-x}$. So $F(x) = x/y^{<-1>} = x/(1-e^{-x})$.

9. (Optional) A tree on $\{0, 1, \ldots, n\}$ is called *alternating* if for every vertex i all neighbors are either greater than i, or all are smaller than i. Let h_n be the number of alternating trees on $\{0, 1, \ldots, n\}$.

   (a) List all the alternating trees for h_2.

   (b) Prove that $H(z) = \sum_{n\geq 0} h_n \frac{z^n}{n!}$ satisfies the equation
$$H(z) = e^{\frac{z}{2}(H(z)+1)}.$$

   Solution. Removing the vertex 0 from an alternating tree on $\{0, 1, \ldots, n\}$, one gets a forest of rooted alternating trees, where the root is a vertex whose neighbors are greater than it. To count the number of rooted alternating trees with the root being a "local minimal", note that if the vertex v is a local minimal in an alternating tree T, then it is a local maximal when we re-label vertex i by $n - i$, for each i. Thus the total number of such rooted trees on n vertices is $nh(n-1)/2$, for $n \geq 2$. For $n = 1$, it is 1.

   Therefore,
$$h(n) = \sum_{\pi = B_1,\ldots,B_k \in \Pi_n} f(\#B_1)f(\#b_2)\cdots f(\#B_k),$$
   where $f(n) = nh(n-1)/2$ for $n \geq 2$, and $f(1) = 1$. By the exponential formula,
$$H(z) = \exp(\sum_{n\geq 1} h(n)\frac{z^n}{n!}) = \exp(\frac{z}{2}(H(z)+1)).$$

   (c) Deduce from above that
$$h_n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(k+1)^{n-1}.$$

   Solution. Set $A(z) = z(H(z)+1)$. Then the equation in part (a) becomes $\frac{A}{z} - 1 = e^{A/2}$, so $A = z(1 + e^{\frac{A}{2}})$. Thus
$$A^{<-1>} = \frac{z}{1+e^{z/2}}.$$

By Lagrange

$$
\begin{aligned}
[z^{n+1}]A &= \frac{h_n}{n!} = \frac{1}{n+1}[z^n](1+e^{z/2})^{n+1} \\
&= \frac{1}{n+1}[z^n]\sum_k \binom{n+1}{k} e^{kz/2} \\
&= \frac{1}{n+1}[z^n]\sum_k \binom{n+1}{k} \sum_i \frac{k^i}{2^i}\frac{z^i}{i!} \\
&= \frac{1}{n+1}\frac{1}{2^n n!}\sum_{k=1}^{n+1}\binom{n+1}{k} k^n \\
&= \frac{1}{2^n n!}\sum_{k=0}^{n}\binom{n}{k}(k+1)^{n-1}.
\end{aligned}
$$

The claim follows.