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Calculus[1]

The invention of Calculus must be considered one of the greatest achievements of mankind. Though it, all of dynamical physics, electricity, magnitism, and most of the other scientific aspects of our life, have been derived and explained. Because of it, many ancient superstitions have been debunked. Because of its success, the rigor with which it was created has become the *lingua franca* of all of science — and science has profited from these demands. The results have been profound; what is more, the results continue to flood our days with new discoveries with no end in sight.

1 Isaac Newton

By some accounts **Isaac Newton** (1643 - 1727) was born on Christmas Day in 1642. This is accounted for by calendar reform issues. Born in the small town of Woolsthorpe, Newton was the only son of a local yeoman and Hannah Ayscough. His father died three months before his birth. He was a tiny, frail baby and was not expected to survive his first year. His mother, concerned about her welfare, married again, this time to the well-to-do minister Barnabas Smith. When Isaac was just three, Hannah her baby with his grandmother, partly at the insistence of Rev. Smith. Smith, who was of advanced years, had a previous family and Hannah fully expected to become his widow within due course. However, plans do not always work out, and by Smith, she bore a a son and two daughters. It was only after nine more years, when Smith passed away, that Isaac was reunited with his mother. However, relations between the two were permanently strained. Indeed, his well noted psychotic tendencies have been ascribed to this traumatic event. He never married.

Newton was sent to school at Grantham, where he resided with the local apothecary, Mr. Clark. It was there that he began his fascination with pharaceutical remedies. He learned to make his own remedies and often made them for others. Thoughout his life Newton was a hypochondriac. He was gifted with his hands making sundials, kites with lanterns and other rather well crafted devises. Though he loved books, he was a self-confessed inattentive student. However he would study just before examinations and score better grades than his peers.

He was mostly a solitary child, given to self-study. He kept various notebooks in which he copied long passages from books he was reading. He also kept something of a diary in which he maintained a very private life, confessing at times contempt and loathing for various people. He even wished death upon another. Though his mother tried hard to make a farmer of the young Newton, it was not to be. After a second visit to Hannah, his teacher Stokes at last convinced her to send him to Cambridge, some fifty miles to the north. He was not popular among the help on the farm, and they were delighted when he left.

At this time, the Aristotelian view held sway in physics. His syllogism based rigidity distorted the world and confined thinkers from a realistic fact based model of the universe. So compelled were scientists to the Aristotelian straight jacket, scientists could hardly perform experiments and make conjecture on their basis. Aristotle's dominance left little room for alternative ideas. Aristotle's dogma was like a religion to his followers, passed down from one generation to the next. The legacy of ancient Greece was still very powerful.

When Newton arrived in Cambridge, he found a backwater town with beggars and thieves everywhere. Most people, about five thousand in all, were illiterate. About 3000 students attended Cambridge at the time. The school itself was academically backward. Newton arrived with 15 pounds tuition and 10 pounds expense money. Though Hannah received about 700 pounds annually, she kept her son at near poverty. As one of the lower class students, he had to take a job as a subsizar. This meant that he was a servant to more privileged students, running errands, cleaning bedrooms, and emptying bedpans. He appreaciated thriftiness and earned money by loaning others the little he had.

The greatest scientist of the second millennium began college humbly.

As he was about two years older than his class, he felt more out of place than he might have. In consequence, he isolated himself from other students. His beliefs were deeply puritanical, and like, for example John Napier, he distrusted Catholics his whole life. This, in spite of the fact that the end of Cromwell's regime brought a general enlightenment.

Having bought a prism at a local Stourbridge Fair, he began his first experiments in optics in 1664. Eventually, he published his results fourty years later, in 1704. The prevailing theory of the day was

Descartes's ether theory. In his experiments, Newton corrrectly assessed the nature of color.

In 1665, Cambridge was closed for two years on account of the black plague. Newton returned home to Woolsthorpe. It was there he began his studies that would lead to his first mathematical and scientific breakthoughs. Though Newton himself reportedly recalled that his ideas of universal gravitation attraction was inspired by the falling of an apple from a tree, this story is generally regarded as a fabrication. It is important to recognize that although Newton had remarkable insights and ideas, it took him many years to refine them and many more to publish them. Newton was quite sensitive to any form of criticism and more so to any predations of his character.

In 1664, Newton took his examinations and his examiner was Isaac Barrow. Barrow, it is believed, fully understood the potential of the young student before him and passed Newton in spite of the fact that he had not read Euclid. Overall, Newton did little by way of formal study, graduating eventually but with an undistinguised career.

Newton, as we know, laid the foundation for differential and integral calculus. This alone would establish him as one of the great thinkers of all time. That he developed calculus in the service of the theory of gravitation and motion make it all the more remarkable. Even his work on optics would place him among the greatest scientists. Newton, like perhaps only Archimedes and Aristotle before him', was a person off the scale of normal genius. He was one whose "shaped the categories of the human intellect". It is not possible to measure Newton in any ordinary sense.

If he had not invented calculus – as he is ascribed to have done – he would still be one of the great thinkers of all time.

His career included contributions to:

- **Optics** – central activity of the scientific revolution. (Descartes also made contributions here.) He denied the homogeneity of light, stating that it was complex and heterogeneous.
- **Planetary Motion.** *Philosophiae Naturalis Principia Mathematica* 1687. The fundamental work of modern science. It includes planetary motion and **Universal Gravitation**.

In the opening sections of the *Principia* Newton had so generalized and clarified Galileo's ideas on motion that ever since we refer to them as "Newton's laws of motion."

Then Newton went on to combine these laws with Kepler's laws and with Huygens law of centripetal motion to establish the unifying principle in the universe that any two particles attract each other according as the inverse square law of distance.

This had been anticipated by Robert Hooke as well as Edmund Halley. But Hooke's concepts were intuitive. Newton convinced the world by carrying off the mathematics needed for the proof.

In 1693 Newton has a nervous breakdown, after which he substantially retired from research.

He was also Master of the Mint following the publication of the *Principia*. He took an active interest in his duties and became the scourge of counterfeiters, sending many to the gallows.

In 1703, he was elected president of the Royal Society and assumed the role of patriarch of English science. In 1705 (08?) he was knighted, the first scientist so honored.

Over the years he had furious debates with other scientists, notably Robert Hooke and John Flamsteed.

It is generally agreed that Newton developed calculus before **Gottfried Wilhelm Leibnitz** seriously pursued mathematics. It is also agreed that Leibnitz developed it independently. Leibnitz published in 1684.

A fracas of priority of discovery developed into a small war. Newton was drawn in; and once his temper was triggered by accusations of dishonesty, his anger was beyond constraint. Leibnitz's conduct though not pleasant, paled beside that of Newton. Said his assistant Whiston:

> Newton was of the most fearful, cautious and suspicious temper that I ever knew.

Newton's mathematical works include:

- power series – binomial theorem
- fluxions – calculus and the fundamental theorem
- applications of fluxions to extrema problems, area problems
- algorithms for the use of calculus
- a concept of limit.

1.1 Newton's mathematics

Newton's work on the **binomial theorem** is nothing short of remarkable. He begins, as did Wallis, by making area computations of the curves $y = (1 - x^2)^n$, and tabulating the results. He noticed the Pascal triangle and reconstructed the formula

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

for positive integers n.

Now to get to compute $\int_0^x \sqrt{1-x^2}\,dx$, i.e. $n = 1/2$, he simply applied this relation with $n = 1/2$. This of course generated an infinite series because the terms do not terminate.

Next he generalized to function of the form $y = (a + bx)^n$ for any n. This gave him the general binomial theorem – but not a proof.

He was able to determine the power series for $\ln(1+x)$ by integrating the series for $(1 + x)^{-1}$, written according as the binomial series. In modern notation, we have

$$\begin{aligned} (1+x)^{-1} &= 1 + \binom{-1}{1}x + \binom{-1}{2}x^2 + \binom{-1}{3}x^3 + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots . \end{aligned}$$

Now integrate to get the series

$$\log(1 + x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots .$$

With this he was able to compute logarithms of the number 1 ± 0.1, $1 + \pm 0.2$, 1 ± 0.01, $1 + \pm 0.02$ to 50 places of accuracy. Then using identities such as

$$2 = \frac{1.2 \times 1.2}{0.8 \times 0.9}$$

he was able to compute the logarithm of many numbers.

Next he worked out the power series for $y = \arcsin x$, and ultimately found the power series for $y = \sin x$ using his method of *affected equations*. The reason for this apparent reversal of what we would think to be the order of discovery is that

$$y = \arcsin x = 2\int_0^x \sqrt{1-x^2}\,dx - x\sqrt{1-x^2}$$

Thus the binomial series and integration term-by-term could be applied.

The confirmations he achieved using his power series method justified in his mind the ultimate correctness of this procedure. But convergence?

Newton was unconcerned with questions of convergence.

Newton developed algorithms for calculating fluxions defined in modern terms as

$$\begin{aligned} \dot{x} &= \text{fluxion} \\ x = x(t) &= \text{fluent} \end{aligned}$$

to solve the problems:

- Find the speed of notion of any fluent.
- Given the speed find the length of space at any time t.

He assumes a form $f(x, y) = 0$ and produces the differential equation

$$\frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt} = 0,$$

using the procedure of Hudde. His method builds into it the product rule for derivatives.

He justifies this rule by defining the **moment**

$$x + o\dot{x},$$

substituting and resolving the terms àla Fermat. Note the term o is viewed as infinitely small.

At this time infinitesimals have been completely accepted by some while wholy rejected by other. That is, the infinitesimal is a *real* object, not a potentiality or convenience of expression!!!!

There is, I must emphasize, no theory of any of this infinitesimal analysis. Mathematicians are "flying about by the seat of their pants", just doing it, and not all worried about the grand Aristotelian/Euclidean plan.

To resolve the "length of space" question, Newton reverses the procedure if possible. This is an antiderivative approach. Otherwise he resorts to power series.

Example. Consider the equation

$$\dot{y}^2 = \dot{x}\dot{y} + x^2\dot{x}$$

is resolved as

$$\begin{aligned}\left(\frac{\dot{y}}{\dot{x}}\right)^2 &= \frac{\dot{y}}{\dot{x}} + x^2 \\ \frac{\dot{y}}{\dot{x}} &= \frac{1}{2} \pm \sqrt{\frac{1}{4} + x^2}.\end{aligned}$$

Applying the binomial theorem we get for the plus root

$$\frac{\dot{y}}{\dot{x}} = 1 + x^2 - x^4 + 2x^6 - 5x^8 + \cdots \ .$$

Hence one solution is

$$y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7 \cdots .$$

The other is determined similarly.

Newton discovered a method for finding roots of equations which is still used today.

Among the curves worked on by Newton were the Cartesian ovals, the Cissoid, the Conchoid, the Cycloid, the Epicycloid, the Epitrochoid, the Hypocycloid, the Hypotrochoid, the Kappa curve and the Serpentine. Newton gave a classification of cubic curves.

Newton gives methods of finding extrema problems normals, tangents and areas.

The concept of limit appears in the *Principia* as the "ultimate ratio of evanescent quantities" which is similar to our own notion of limit of a difference quotient. He goes to some effort to assuage the great bulk of mathematicians still wedded to Greek geometry and thought.

By studying the finest work of the time Newton was led to important new syntheses. To develop them fully he acquired a mastery of analytical techniques unsurpassed in his time. Thus he was able to derive simple and general methods compared with the laborious work of his contemporaries. Newton thought analytically **in the modern sense**. This was an enormous advantage.

2 Gottfried Wilhelm von Leibniz

On July 1, 1646, Leibniz (1646 - 1716) was born into a pious Lutheran family. He was educated at the Nicolai School. Though his father died when he was just six years old, much of his education came from his father's library. At the age of fifteen, he entered the University of Leipzig as a law student. It was at the University he encountered for the first time the great masters of science such as Galileao, Francis Bacon, Thomas Hobbes, and René Descates. Among his goals, even at an early age and extending thoughout his adult life, was to "reconcile" these thinkers with Aristotle and the Scholastics.

He was a deep thinker from the onset of his career. His baccalaureate thesis of 1663, *De Principio Individui* ("On the Principle of the Individual"), emphasized the existential value of the individual, who is not to be explained either by matter alone or by form alone but rather by his whole being (*entitate tota*). He received his doctorate in 1667. His original thesis idea was ambitious, to work out an algebra of human thought, an attempt to symbolize thought and to work out a combinatorial calculus.

Though a philosopher and mathematician his entire life, he believed that academics should be founded in a wide variety of arts. Toward this end, he worked on hydraulic presses, windmills, lamps, submarines, clocks, and a variety of mechanical devices. He also experimented with phosphorus, developed a water pump run by windmills, which aided in the exploitation of the mines of the Harz Mountains. Indeed he frequently worked these mines as an engineer from 1680 to 1685.

In 1672, on a diplomatic mission to Paris, Leibnitz met and for the first time studied mathematics seriously with Huygens. As a diplomat he made two trips to London, in 1673 and 1676, where it is possible he had access to Newton's manuscript. Only ten years later he began to publish short pieces on calculus.

By 1685, Leibniz had worked out the foundations of both integral and differential calculus. With this discovery, he ceased to consider time and space as substances–another step closer to monadology. He began to develop the notion that the concepts of extension and motion contained an element of the imaginary, so that the basic laws of motion could not be discovered merely from a study of their nature.

Always conscious of the presentation of an idea, he developed the present day notation for the differential and integral calculus. He never thought of the derivative as a limit.

Leibniz founded the Berlin Academy in 1700 and was its first president. He became more and more a recluse in his later years.

2.1 Leibnitz' Mathematics

His first investigations were with the **harmonic triangles** H.

$$\begin{array}{cccc}
\frac{1}{1} & & & \\
\frac{1}{2} & \frac{1}{2} & & \\
\frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \\
\frac{1}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{4} \\
\vdots & & &
\end{array}$$

From this he noticed that

$$\begin{aligned}
H_{ij} &= H_{i-1,j} - H_{i,j+1} \\
H_{ij} &= H_{i,j-1} - H_{i-1,j-1}.
\end{aligned}$$

This means that sums along 45° diagonals of H are sums of differences. So for example

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots = 1.$$

Also,

$$\begin{aligned}
\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \cdots &= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{12}\right) + \left(\frac{1}{12} - \frac{1}{20}\right) + \cdots \\
&= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots \\
&\quad - \left(\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots\right) \\
&= 1 - \left(1 - \frac{1}{2}\right) = \frac{1}{2}.
\end{aligned}$$

Multiplying by 3 we sum the pyramidal numbers

$$1 + \frac{1}{4} + \frac{1}{10} + \cdots = \frac{3}{2}.$$

The importance of these ideas rested with their applications of summing differences in geometry. That is, he sees the possibility

$$d \int y = y \qquad \int dy = y$$

where

$$\int y = y_0 + y_1 + \cdots + y_n$$
$$dy = y_i - y_{i-1}$$
$$\sum dy \doteq \int dy = y$$

Leibnitz interpreted the term $d \int y$ as area

$$d \int y = y\, dx$$

(i.e. $\frac{d}{dx} \int y = y$). This gives in principle his fundamental theorem.

By 1673 he was still struggling to develop a good notation for his calculus and his first calculations were clumsy. On 21 November 1675 he wrote a manuscript using the $\int f(x)\, dx$ notation for the first time. The $\int$ symbol was an *elongated* S, which of course stood for sum.

In the same manuscript the product rule for differentiation is given. The quotient rule first appeared two years later, in July 1677. **Leibnitz was very conscious of notation.** He recognizes two separate branches.

differentia and *summa*

Leibnitz' clarity of differencing was applied to the difference triangle, which is the one we use today. From it he derives the sum, product and quotient rules, at first erroneously. It is

$$d(xy) = x\, dy + y\, dx$$

and **not**

$$d(xy) = dx\, dy$$

as he originally thought.

In 1684 he gives the power rules for powers and roots. The chain rule is transparent from his notation

$$\begin{aligned} d(x^n) &= nx^{n-1}dx \\ d(\sqrt[b]{x^a}) &= \frac{a}{b}\sqrt[b]{x^{a-b}}dx \end{aligned}$$

In 1684 he solves a problem posed by Debeaune to Descartes in 1639, that being to find a curve whose subtangent is a constant:

$$y\frac{dx}{dy} = a \quad \text{or} \quad a\,dy = y\,dx$$

Leibnitz takes $dx = 1$ and gets $y = k\,dy$; that is, the ordinates are proportional to their increments. So the curve is logarithmic ("exponential" in modern terms).

In 1695, he computes the differential of $z = y^x$ where y and x are variables. With Jacques Bernoulli's suggestion he solves this by taking the logarithm of both sides.

$$\begin{aligned} \log z &= x \log y \\ \frac{dz}{z} &= x\frac{dy}{y} + \log y dx. \end{aligned}$$

Hence

$$d(y^x) = xy^{x-1}dy + y^x \log y dx$$

Leibnitz develops a fundamental theorem: One can find a curve z such that $dz/dx = y$. It is given by

$$\int_0^b y\,dx = z(b).$$

By 1690 Leibnitz has discovered most ideas in current calculus text books.

Leibnitz was more interested in solving differential equations than finding areas. Among them he derives and solves the familiar differential equation for the *sine* function. He developed the *separation of variables* method.

Among the curves worked on by Leibniz were the Astroid, the Catenary, the Cycloid, the Epicycloid, the Epitrochoid, the Hypocycloid, the Hypotrochoid, the semi cubical parabola and the Tractrix.

3 Summary

Our modern calculus resembles that of Leibnitz far more than Newton. Possibly because of Newton's reluctance to publish Leibnitz's version became better known on the continent. Leibnitz's calculus was somewhat easier to comprehend and apply. This cost English mathematics almost a century of isolation from the continent and the resulting progress implied.

4 First Calculus Texts:

- L'Hospital, *Analyse des Infiniment Petits four l'intelligence des lignes courbes,* 1696 He makes fundamental statements in the beginning of his text that make clear that he assumes infinitesimals are real objects, though arbitrarily small.
- Humphrey Ditton (1675-1715) *An Institution of Fluxions*, 1706
- Charles Hayes (1678-1760) *A Treatise on Fluxions*, 1706.