

Eudoxus of Cnidus¹

Eudoxus (c. 400 BCE), son of Aeschines, is ranked among the greatest of the ancient mathematicians, surpassed perhaps only by Archimedes — but more on Archimedes later. The few facts known concerning his life are derived substantially from the histories written by Diogenes Laërtius in the 3rd century BCE.

- Eudoxus was born in Cnidus, on the Black Sea. Eudoxus learned mathematics and medicine at a school that rivaled that of Hippocrates of Cos. A well-to-do physician, very much impressed by his ability, paid his way to Athens so that he could study at Plato's Academy (est. 387). He also spent 16 months in Egypt during the reign of Nectanebo I (380-363). At Heliopolis, now a Cairo suburb, Eudoxus learned the priestly wisdom, which included astronomy.
- He studied mathematics with Archytus (a Pythagorean) in Tarentum.
- He studied medicine with Philistium on Sicily.
- At the age of 23 he went to Plato's academy in Athens to study philosophy and rhetoric.
- He established a school having many pupils at Cyzicus on the sea of Marmora (Marmara) in what is now Balikhisar, Turkey. (Balikhisar, in a position of strategic commercial importance, was likely founded as a colony of Miletus in 756 BCE.)
- In 365 B. C. he returned to Athens with his pupils. He became a colleague of Plato.
- At the age of 53 he died in Cnidos, highly honored as a law-giver/legilator.
- He was the leading mathematician and astronomer of his day.

Eudoxus was the most reknown astronomer and mathematician of his day. In astronomy devised an ingenious planetary system based on spheres.

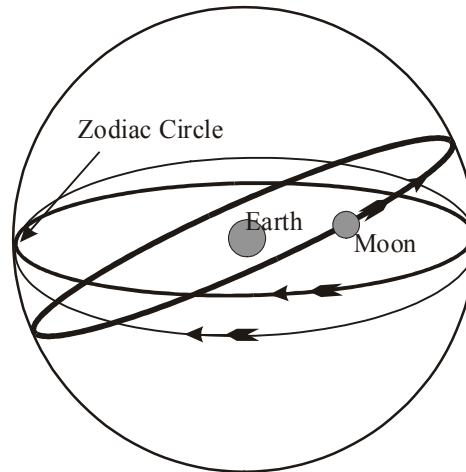
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- ★ The spherical earth is at rest at the center.
- ★ Around this center, 27 concentric spheres rotate.
- ★ The exterior one carries the fixed stars,
- ★ The others account for the sun, moon, and five planets.
- ★ Each planet requires four spheres, the sun and moon, three each.

Consider the moon.

- The outer sphere rotates in one day as the sphere of the stars and with axis perpendicular to the zodiac circle. One period is 24 hours.
- The next middle sphere rotates on a circle at an angle to the plane of zodiac circle, and from east to west. One period is 223 lunations. From this sphere the “recession of the nodes” is realized.
- The inner sphere rotates about an axis inclined to the axis of the second at an angle equal to the highest latitude attained by the moon, and from west to east. The draconitic month, the period of this sphere, is 27 days, 5 hours, 5 minutes. The moon is fixed on the great circle at this angle.

Homocentric spheres for the moon



★ The description of the motion of the planets is more clever still.

★ This model was improved by Callippus of Cyzicus (fl. 370 BCE), a student of Eudoxus, who added spheres to improve the theory, especially for Mercury and Venus and by Aristotle who added to this “retrograde” spheres. Aristotle’s cosmos, modelled like an onion, consisted of a series of some 55 spheres nested about a center, the Earth. However, these emendations never accounted for variation of luminosity, which had been observed and bounded elongations (e.g. Venus is never observed to be more than about 48° and Mercury never more than about 24° from the Sun).

★ Eudoxus also described the constellations and the rising and setting of the fixed stars.

★ Within 50 years the whole theory had to be abandoned.

Eudoxus’s contributions to mathematics include:

- A theory of proportion; this allowed the study of irrationals (incommensurables)².
- The concept of magnitude, as **not** a number but stood for such as line segments, angles, areas, etc, and which could vary continuously. Magnitudes were opposed to numbers, which could change discontinuously. This avoided giving numerical values to lengths, areas, etc. Consequently great advances in geometry were made³.
- The method of exhaustion.
- Establishing rigorous methods for finding areas and volumes of curvilinear figures (e.g. cones and spheres).
- A profound influence in the establishment of *deductive organization* of proof on the *basis of explicit axioms*.

There is little question that Eudoxus added to the body of geometric knowledge. Details are scant, but probably his main contributions can be found in Euclid, Books V, VI, and XII.

The Theory of Proportion of Eudoxus is found as Definition 5 of Euclid, Book V.

Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

In modern terms: $a/b = c/d$ if and only if, for all integers m and n , whenever $ma < nb$ then $mc < nd$, and so on for $>$ and $=$.

²Incommensurables had only been recently discovered, reportedly by Hippasus of Metapontum

³at the expense of algebra. All mathematicians were driven to geometry. The consequence persists. Even today, we still speak of x^2 and x^3 as *x squared* and *x cubed*.

This is tantamount to an infinite process. But it was needed to deal with incommensurables.

The Method of Exhaustion unquestionably helped resolve number of loose ends then extant. It contained as Proposition 1 of Book X.

Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let $a > \epsilon > 0$ be given⁴. Let $a > s_1 > a/2$, and $a_1 = a - s_1$. Let $a_1 > s_2 > a_1/2$ and $a_2 = a_1 - s_2$. Continue this process, generating the sequence a_1, a_2, \dots , we eventually have $a_n < \epsilon$.

How does this differ from our limit concept today?

With this result, Eudoxus was able to establish following:

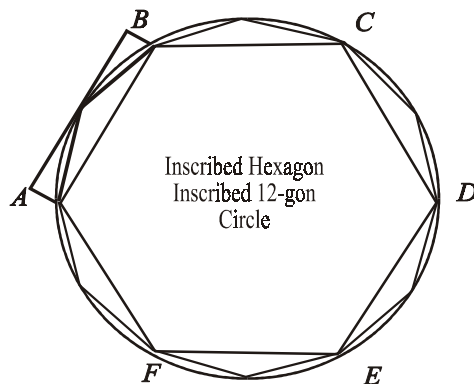
Proposition 2. (Book XII) Circles are to one another as the squares on the diameters.

This was proved on the basis of the previous proposition.

Proposition 1. (Book XII) Similar polygons inscribed in circles are to one another as the squares on the diameters.

⁴Note the assumption that $\epsilon > 0$ is superfluous in the language of the time as there were only positive numbers. So, the assumption is implicit

To prove the Proposition 2, polygonal figures, of indefinitely increasing numbers of sides, are both inscribed and circumscribed in the circle. Assuming Proposition 2 does not hold will lead to the contradiction that the result must be false for the polygons also.



Proof of Proposition 2. Let a and A , d and D be the respectively diameters of the circles. Suppose that

$$\frac{a}{A} > \frac{d^2}{D^2}.$$

Then there is an $a' < a$ so that

$$\frac{a'}{A} = \frac{d^2}{D^2}.$$

Set $\epsilon = a - a'$. Let p_n (resp. P_n) be the inscribed regular polygons of n sides in circle a (resp A). Then

$$a - p_{2n} < \frac{1}{2}(a - p_n).$$

By the method of exhaustion it follows that for large enough n

$$a - p_n < a - a' = \epsilon$$

which implies that

$$p_n > a'.$$

We know that

$$\frac{p_n}{P_n} = \frac{d^2}{D^2} = \frac{a'}{A}.$$

Thus, since $p_n > a'$, it follows that $P_n > A$. But this is impossible, and we have a contradiction.

To complete the proof, it must now be shown that

$$\frac{a}{A} < \frac{d^2}{D^2}$$

is also impossible.

This is a *double reductio ad absurdum* argument, a requirement of this method.

Eudoxus also demonstrated that *the ratios of the volumes of two spheres is as the ratio of the cubes*⁵ of their radii.

On pyramids Eudoxus proved

Proposition 5. (Book XII) Pyramids which are of the same height and have triangular bases are to each other as their bases.

Other propositions are more famous⁶:

Proposition. The volume of every pyramid is one third of the prism of on the same base and with the same height.

Proposition. The volume of every cone is one third of the cylinder on the same base and with the same height.

Curiously, the proof is by the method of slabs, familiar to all college freshmen after completing their first calculus course.

⁵This is a modern word; the original word as given by Archimedes is *triple*.

⁶Try to prove either of these without calculus!! Yet, the Egyptians seems to have been aware of the formulas. Levels of understanding progress.