June 5, 2007

Islamic Mathematics and Mathematicians[1]

1 Introduction

The torch of ancient learning passed first to one of the invading groups that helped bring down the Eastern Empire. Within a century of Muhammad's conquest of Mecca, Islamic armies conquered lands from northern Africa, southern Europe, through the Middle East and east up to India. The empire was immense, rivaling that of Rome itself. Though the Arabs initially focused on conquest, nonetheless to them ancient science became precious treasure. The Qur'an, the sacred book of Islam, praised medicine as an art close to God. Astronomy and astrology were believed to be a pathway to discover God's will. Within a century of that the Caliphate split up into several parts. The eastern segment, under the Abbasid caliphs, became a center of growth, of luxury, and of peace. In 766 the caliph al-Mansur founded his capitol in Baghdad and the caliph Harun al-Rashid, established a library. The stage was set for his successor, Al-Ma'mum.

In the 9th century Al-Ma'mum established Baghdad as the new center of wisdom and learning. He establihed a research institute, the *Bayt al-Hikma* (House of Wisdom), which would last more than 200 years. Al-Ma'mum was responsible for a large scale *translation* project of translating as many ancient works as could be found. Greek manuscripts were obtained through treaties. By the end of the 9th century, the major works of the Greeks had been translated[2]. In addition, they learned the mathematics of the Babylonnians and the Hindus. The Arabs did not stop with assimilation. They innovated and criticized. They absorbed Babylonian and Greek astronomy and constructed large scale astronomical observatories and made measurements against which predictions of Ptolemy could be checked. Numbers, particularly numbers as used in algebra fascinated the Islamic mathematicians. Surely, if one measures Islamic mathematics against the ancients, it would be in algebra where their originality and depth is most clearly evident.

[2] Interestingly, most of the translations were done from Greek and Syriac by Christian scholars, with support Muslim patrons. These included the caliph and wealthy individuals

What follows is a brief introduction to a few of the more prominent Arab mathematicians, and a sample of their work in more or less chronological order. Observe the progression of their mathematics over five century of labor. One should not underestimate the importance of the Islamic world for the preservation of ancient learning.

The illustration below shows the same famous figure from Euclid's *Elements* in ancient Arabic translation.

2 Abu Ja’far Muhammad ibn Musa Al-Khwarizmi(790 - 850)

al-Khawarizmi
Issued by the Soviet Union on Sept. 6, 1983 to mark the 1200th birth anniversary.

Also known as Al-Khwarizmi(which is spelled in several ways), he is no doubt the best known of the Islamic mathematicians, and his work is among the most influential of the islamic school. Indeed, his books were studied long into the Renaissance. By reason of his work on algebra Al-Khwarizmi is sometimes called the *“Father of Algebra”*. Al-Khwarizmi’s most important work *Hisab al-jabr w’al-muqabala* written in 830 gives us the word algebra. This treatise classifies the solution of quadratic equations and gives geometric methods for completing the square. No symbols are used and no negative or zero coefficients were allowed.

Al-Khwarizmi also wrote on Hindu-Arabic numerals. The Arabic text is lost, but a Latin translation, *Algoritmi de numero Indorum*, which in English is *Al-Khwarizmi on the Hindu Art of Reckoning*, gave rise to the word algorithm deriving from his name in the title.

To him we owe the words

Algebra *Algorithm*

His book *Al-jabr wál Mugabala*, on algebra, was translated into Latin and used for generations in Europe.

- It is strictly rhetorical – even the numerals are expressed as words. And it is more elementary than *Arithmetica* by Diophantus.
- It is a practical work, by design, being concerned with straightforward solutions of deterministic problems, linear and especially quadratic.
- Chapters I–VI covers cases of all quadratics with a positive solution in a systematic and exhaustive way.
- It would have been easy for any student to master the solutions. Mostly he shows his methods using examples – as others have done.
- Al-Khwarizmi then establishes geometric proofs for the same solutions of these quadratics. However, the proofs are more in the Babylonnian style.
- He dealt with three types of quantities: the **square** of a number, the **root** of the square (i.e. the unknown), and **absolute numbers.** He notes six different types of quadratics:

$$\begin{aligned} ax^2 &= bx \\ ax^2 &= c \\ bx &= c \\ ax^2 + bx &= c \\ ax^2 + c &= bx \\ ax^2 &= bx + c \end{aligned}$$

- The reason: no negative numbers — no non-positive solutions
- He then solves the equation using essentially a rhetorical form for the quadratic equation. Again note: he considers examples only. There are no "general" solutons.

3 Other Arab mathematicians

We recount here a few of the other prominent Islamic mathematicians. However, it is important to recognise that this is just the tip of the iceberg. Islamic mathematics and mathematicians is a very active area of mathematics, one that will reveal much much more that the mere sketch we have today.

3.1 Shuja ibn Aslam ibn Muhammad ibn Shuja (c. 850 - 930)

Abu Kamil Shuja, an Egyptian sometimes known as al'Hasib, worked on integer solutions of equations. He also gave the solution of a fourth degree equation and of a quadratic equations with irrational coefficients. Abu Kamil's work was the basis of Fibonacci's books. He lived later than Al-Khwarizmi; his biggest advance was in the use of irrational coefficients (surds).

3.2 Abu'l-Hasan Thabit ibn Qurra (826 - 901)

Thabit, a native of Harran, inherited a large family fortune which enabled him to go to Baghdad where he obtained his mathematical training. Returning to Harran, he promoted liberal philosophies which led to a religious court appearance. He was forced to recant his 'heresies'. To escape further persecution he left Harran and returned to Baghdad where he was appointed court astronomer. Thabit generalized Pythagoras's theorem to an arbitrary triangle, as did Pappus. He also considers parabolas, angle trisection and magic squares. He was regarded as Arabic equivalent of Pappus, the commentator on higher mathematics.

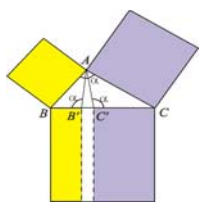
He was also founder of the school that translated works by Euclid, Archimedes, Ptolemy, and Eutocius, but Diophantus was unknown to the Arabs until the 10^{th} century. He also translated Nicomachus' *Arithmetics*. Without his efforts many more of the ancient books would have been lost.

Perhaps most impressive is his contribution to **amicable** numbers. Recall, two numbers are called amicable if each number is the sum of the set of proper divisors of the other. Thabit discovered a beautiful rule for finding amicable numbers.

Following Thabit, Kamal ad-Din al-Farisi (d. c. 1320) gave the pair 17,296 and 18,416 as an example of Thabit's rule, and in the 17^{th}

century Muhammad Baqir Yazdi gave the pair 9,363,584 and 9,437,056.

Theorem. (Generalization of Pythagorean Theorem.) From the vertex A of ΔABC, construct B' and C' so that $\angle AB'B = \angle AC'C = \angle A$ Then $|AB|^2 + |AC|^2 = |BC|(|BB'| + |C'C|)$



Proof. Apply similarity ideas

$$\frac{|AB|}{|BC|} = \frac{|BB'|}{|AB|} \Rightarrow |AB|^2 = |BC|\,|BB'| \ldots \text{ etc.}$$

Note: If $\angle AB'B = \angle AC'C = 90°$, this is the Pythagorean Theorem. This is our second generalization of the Pythagorean Theorem.

3.3 Abu l'Hasan al-Uqlidisi, c.950

In al-Uqlidisi's book *Kita b al-fusul fi-l-hisab al-Hindii* (*The book of chapters on Hindu Arithmetic*), two new contributions are significant: (1) an algorithm for multiplication on paper is given, and (2) decimal fractions are used for the first time. Both methods do not resemble modern ones, but the methods are easily understood using modern terminology.

3.4 Mohammad Abu'l-Wafa al'Buzjani

Mohammad Abu'l-Wafa al'Buzjani, (940 - 998) was born in Buzjan (now in Iran) in Baghdad (now in Iraq). Abu'l-Wafa translated and wrote commentaries, since lost, on the works of Euclid, Diophantus and Al-Khwarizmi. In so doing, for example, he translated *Arithmetica* by Diophantus. Abu al-Wafa' also wrote *Kitab fima yahtaj ilayh al-kuttab wa al-ummal min 'ilm al-hisab* ("Book on What Is Necessary from the Science of Arithmetic for Scribes and Businessmen") and *Kitab fima yahtaj ilayh al-sani 'min al-a'mal al-Handasiyha* ("Book on What Is Necessary from Geometric Construction for the Artisan").

He worked in an observatory in Baghdad, where he built the first wall quadrant for observing stars. In his work on lunar theory he utilized the tangent and cotangent functions. He also invented the secant and cosecant functions, and divised new methods for computing the sine table. His tables of sines and tangents were compiled to 15' intervals. Moreover, his trigonometric tables are accurate to **8 decimal places** (converted to decimal notation) while Ptolemy's were only accurate to 3 places!!

3.5 Abu Bakr al-Karaji, early 11$^{\text{th}}$ century

Known more familiarly as al-Karkhi, he was an Arabic disciple of Diophantus – without Diophantine analysis. He gave numerical solution to equations of the form

$$ax^{2n} + bx^n = c$$

(only positive roots were considered).

He proved

$$1^3 + 2^3 + \cdots + 10^3 = (1 + 2 + \cdots + 10)^2$$

in such a way that it was extendable to every integer. The proof is interesting in the sense that it uses the two essential steps of **mathematical induction.**[3] Nevertheless, this is the first known proof. al-Karkhi's mathematics, more than most other Arab mathematics, pointed to the direction of Renaissance mathematics.

[3] There is in Pappus' *The Collection* a treatment of a variant to the Shoemaker's knife problem that has been argued to contain the first proof containing the two elements of induction.

3.6 Omar Khayyam (1048 - 1122)

Omar Khayyam's full name was Abu al-Fath Omar ben Ibrahim al-Khayyam and was born in what is now modern Iran. A literal translation of his name means 'tent maker' and this may have been his fathers trade. Khayyam is best known as a result of Edward Fitzgerald's popular translation in 1859 of nearly 600 short four line poems, the *Rubaiyat*.

His brilliant contributions were continued in the 19th century. Among many others, he worked on the issues surrounding the parallel postulate. Using the quadrilateral, he discovered an approach to the investigation that became standard. He discovered exactly what must be showed to prove the parallel postulate, and it was upon these type of ideas that non-Euclidean geometry was discovered. Khayyam also argued that rational numbers should be encompassed as numbers, departing from the Greek tradition, whose influence was then and was to remain a powerful force in mathematics and philosophy until the 19th century.

He also discovered methods of root extraction to an arbitrarily high degree. He discovered (in *Algebra*) a geometrical method to solve cubic equations by intersecting a parabola with a circle but, at least in part, these methods had been described by earlier authors such as Abu al-Jud. To see the construction, consider the circle and parabola

$$(x-a)^2 + y^2 = a^2 + c^2 \qquad y = x^2 + bx + c;$$

Substitute and simplify to get

$$x(x^3 - 2bx^2 - x - 2cx - xb^2 + 2a - 2cb),$$

which factored gives

$$x(x^3 + 2bx^2 + (1 + 2c + b^2)x + 2cb - 2a) = 0.$$

So, the intersection x is the solution of the cubic:

$$x^3 + 2bx^2 + (1 + 2c + b^2)x + 2cb - 2a.$$

Khayyam was an outstanding mathematician and astronomer. His work on algebra was known throughout Europe in the Middle Ages, and he also contributed to a calendar reform. Khayyam refers in his algebra book to another work of his which is now lost. In that lost work Khayyam discusses Pascal's triangle but the Chinese may have discussed triangle slightly before this date.

The algebra of Khayyam is geometrical, solving linear and quadratic equations by methods appearing in Euclid's *Elements*.

Khayyam also gave important results on ratios giving a new definition and extending Euclid's work to include the multiplication of ratios. He poses the question of whether a ratio can be regarded as a number but leaves the question unanswered.

Khayyam's fame as a poet has caused some to forget his scientific achievements which were much more substantial. Versions of the forms and verses used in the *Rubaiyat* existed in Persian literature before Khayyam, and few of its verses can be attributed to him with certainty.

Ulugh Beg
Issued by the Soviet Union
on Oct. 8, 1987

3.7 Ghiyath al'Din Jamshid Mas'ud al'Kashi (1390 - 1450)

al'Kashi worked at Samarkand, having patron Ulugh Beg.[4] He calculated π to 16 decimal places and considered himself the inventor of

[4]Ulugh Beg was the grandson of the Asian (Mongol) conquerer Timur. His father captured the city of Samarkand and gave it to Ulugh Beg. He was primarily an astronomer and he built an observatory, the construction beginning in 1428. In his observations he discovered a number of errors in the computations of Ptolemy. He compiled tables of sines and tangents at one minute intervals. These displayed high accuracy, being correct to at least 8 decimal places. Alas, Ulugh Beg's politics were not up to his science and, on his father's death, he was unable to achieve power despite being an only son. He was eventually put to death at the instigation of his own son.

decimal fractions. In fact, he gives 2π correctly as

$$6.2831853071795865$$

which was the best until about 1700.

He wrote *The Reckoners' Key* which summarizes arithmetic and contains work on algebra and geometry. In another work, al'Kashi applied the method now known as fixed-point iteration to solve a cubic equation having $\sin 1^{\mathrm{o}}$ as a root. Generally, for an equation of the form

$$x = f(x)$$

we define the iteration

$$x_{n+1} = f(x_n)$$

where x_0 is some initial "guess". If the iterations converge, then it must be a solution of the equation. Such a method is called a *fixed point iteration*. Another more famous fixed point iteration coming much later is *Newton's Method*

$$x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}.$$

He also worked on solutions of systems of equations and developed methods for finding the n^{th} root of a number – Horner's method today.

[**Note.** This method also appeared in Chinese mathematics in 1303 in the *Su-yüan-yü-chien* (Precious Mirror of the Four Elements)]

Horner's Method Example. Solve

$$x^2 + 252x - 5292 = 0.$$

First determine that a solution lies between $x = 19$ and $x = 20$. Now apply the transformation

$$y = x - 19$$

to obtain

$$y^2 + 290y - 143 = 0.$$

We know there is a root between $y = 0$ and $y = 1$. Thus there are two ways to approximate the solution for y:

1. If $y \approx 0$ then y^2 is even closer to zero, and this term may be taken as zero, giving the approximate solution

$$y = \frac{143}{290}$$

so that $x = 19\,\frac{143}{290}$.

2. We may also factor the equation as

$$\begin{aligned} y^2 + 290y - 143 &= 0 \\ y(y+290) - 143 &= 0. \end{aligned}$$

Letting the y in the parentheses be 1, solve for the other to get hence the approximation

$$y = \frac{143}{291}$$

so that $x = 19\,\frac{143}{291}$.

Clearly the first is slightly too large, while the second is slightly too small. Which should be selected? al'Kashi selects the second, $y = \frac{143}{291}$. Why?

After Al-Kashi, Arabic mathematics closes as does the whole Muslim world. But scholarship in Europe at this time was on the up-swing.