March 21, 2003

Medieval Mathematics[1]

Because much mathematics and astronomy available in the 12^{th} century was written in Arabic, the Europeans learned Arabic. By the end of the 12^{th} century the best mathematics was done in Christian Italy. During this century there was a spate of translations of Arabic works to Latin. Later there were other translations.

$$\begin{gathered} \text{Arabic} \to \text{Spanish} \\ \text{Arabic} \to \text{Hebrew} \quad (\to \text{Latin}) \\ \text{Greek} \to \text{Latin.} \end{gathered}$$

Example. *Elements* in Arabic → Latin in 1142 by **Adelard of Bath** (ca. 1075-1160). He also translated Al-Khwarizmi's astronomical tables (Arabic → Latin) in 1126 and in 1155 translated Ptolemy's Almagest (Greek → Latin) (The world background at this time was the crusades.)

An interesting note is that while mathematics as a research subject was at a low point during the early middle ages, the notion of proof survived and was even reinvigorated by some of the authors and translators. Perhaps the proof, carrying with it a sense of absolute, which transcended other forces of the age, was the solice of the academician.

1 Gherard of Cremona (1114 - 1187)

Gherard's name is sometimes written as Gerard. He travelled to Toledo, Spain to learn Arabic so he could read Ptolemy's Almagest, since no Latin translations existed at that time. He remained there for the rest of his life. Gherard made translations of Ptolemy (1175) and of Euclid from Arabic. Some of these translations from Arabic became more popular than the (often earlier) translations from Greek. In making translations of other Arabic work he translated the Arabic word for sine into the Latin *sinus*, from where our *sine* function comes. He also translated Al-Khwarizmi.

2 Adelard of Bath, (1075 - 1160)

During this period (12th century) the Hindu numerals became known to Latin readers by Adelard of Bath, also known as **Robert of Chester**. Adelard studied and taught in France and traveled in Italy, Syria and Palestine before returning to Bath. He was a teacher of the future King Henry II. Adelard translated Euclid's *Elements* from Arabic sources. The translation became the chief geometry textbook in the West for centuries. He translated al'Khwarizmi's tables and also wrote on the abacus and on the astrolabe. One book, his *Quaestiones naturales* consists of 76 scientific discussions based on Arabic science.

3 Leonardo Pisano Fibonacci (1170 - 1250)

Fibonacci or Leonard of Pisa, played an important role in reviving ancient mathematics while making significant contributions of his own. Leonardo Pisano is better known to us by his nickname *Fibonacci*, which was not given him until the mid-nineteenth century by the mathematical historian Guillaume Libri. He played an important role in reviving ancient mathematics and made significant contributions of his own. with his father, Fibonacci was born in the city-state of Tuscany (now in Italy) but was educated in North Africa where his father held a diplomatic post. He traveled widely recognizing and the enormous advantages of the mathematical systems used in these countries.

Leonardo *Liber abbaci* (*Book of the Abacus*), published in 1202 after his return to Italy, is based on bits of arithmetic and algebra that Leonardo had accumulated during his travels. The title *Liber abbaci* has the more general meaning of mathematics and calculations or applied mathematics than the literal translation of a counting machine. The mathematicians of Tuscany following Leonardo were in fact called

Maestri d'Abbaco, and for more than three centuries afterwards learned from this venerated book. Almost all that is known of his life comes from a short biography therein, though he was associated with the court of Frederick II, emperor of the Holy Roman Empire.

> "I joined my father after his assignment by his homeland Pisa as an officer in the customhouse located at Bugia [Algeria] for the Pisan merchants who were often there. He had me marvelously instructed in the Arabic-Hindu numerals and calculation. I enjoyed so much the instruction that I later continued to study mathematics while on business trips to Egypt, Syria, Greece, Sicily, and Provence and there enjoyed discussions and disputations with the scholars of those places. Returning to Pisa I composed this book of fifteen chapters which comprises what I feel is the best of the Hindu, Arabic, and Greek methods. I have included proofs to further the understanding of the reader and Italian people. If by chance I have omitted anything more or less proper or necessary, I beg forgiveness, since there is no one who is without fault and circumspect in all matters."

The *Liber abbaci* introduced the Hindu-Arabic place-valued decimal system and the use of Arabic numerals into Europe. *Liber abbaci* did not appear in print until the 19^{th} century. A problem in *Liber abbaci* led to the introduction of the Fibonacci numbers and the Fibonacci sequence for which Fibonacci is best remembered today. The *Fibonacci Quarterly* is a modern journal devoted to studying mathematics related to this sequence.

Fibonacci's other books of major importance are *Practica geometriae* in 1220 containing a large collection of geometry and trigonometry. Also in *Liber quadratorum* in 1225 he approximates a root of a cubic obtaining an answer which in decimal notation is correct to 9 places.

3.1 *Liber abbaci*

Features of *Liber abbaci* include:

- a treatise on algebraic methods and problem which advocated the use of Hindu-Arabic numerals. What is remarkable is that neither

European nor Arab businessmen use these numerals in their transactions, and when centuries later they caught on in Europe, it was the Europeans that taught the Arabs of their use.

- used the horizontal bar for fractions.
- in fractions though the older systems of unit and sexagesimal were maintained!
- contained a discussion of the now-called *Fibonacci Sequence* – inspired by the following problem:

"How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on."

The sequence is given by

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots, u_n, \ldots$$

which obeys the recursion relation

$$u_n = u_{n-1} + u_{n-2}$$

- Some of Fibonacci's results:

Theorem. (i) Every two successive terms are relatively prime.

(ii) $\lim_{n\to\infty} u_{n-1}/u_n = (\sqrt{5}-1)/2$.

Proof. (i) $u_n = u_{n-1} + u_{n-2}$. If $p|u_n$ and $p|u_{n-1}$, then $p|u_{n-2} \Rightarrow p|u_{n-3} \ldots \Rightarrow p|u_1$. #.

(ii) From $u_n = u_{n-1} + u_{n-2}$ we have

$$\begin{aligned} 1 &= \frac{u_{n-1}}{u_n} + \frac{u_{n-2}}{u_n} \\ &= \frac{u_{n-1}}{u_n} + \frac{u_{n-2}}{u_{n-1}} \cdot \frac{u_{n-1}}{u_n}. \end{aligned}$$

So, if $\lim \frac{u_{n-1}}{u_n}$ exists and equals r, it follows that

$$1 = r + r^2 \quad r = \frac{-1 \pm \sqrt{5}}{2} \to \frac{\sqrt{5}-1}{2}.$$

This is a *golden section connection*. To show that u_{n-1}/u_n converges, define $s_n = u_{n-1}/u_n$. Clearly, $u_{n-1}/u_n \geq \frac{1}{2}$. Then

$$1 = s_n(1 + s_{n-1}) \quad \text{or} \quad s_n = \frac{1}{1 + s_{n-1}}$$

$f(x) = \frac{1}{1+x}$, $f'(x) = \frac{-1}{(1+x)^2}$. If $x > 0$ $|f'| < 1$

$$\begin{aligned} |s_n - s_{n-1}| &= |f(s_{n-1}) - f(s_{n-2})| \\ &< f'(\alpha_n)|s_{n-1} - s_{n-2}| < \beta_n|s_{n-1} - s_{n-2}|. \end{aligned}$$

Since $\beta_n < k < 1$, this establishes convergence.

Alternatively: $s_{n+2} = \frac{1}{1+\frac{1}{1+s_n}}$. This is a decreasing sequence because $s_0 > \frac{\sqrt{5}-1}{2}$. So $s_0 > s_2 > s_4 > \cdots$. Now give a lower bound using $s_1,\ s_2, | > \ldots$. Next show that the limits must be the same. etc,etc,etc.

Other properties:

$$\begin{aligned} u_1 &= u_3 - u_2 \\ u_2 &= u_4 - u_3 \\ u_3 &= u_5 - u_4 \\ &\vdots \\ u_{n-1} &= u_{n+1} - u_n \\ u_n &= u_{n+2} - u_{n+1}. \end{aligned}$$

So to get

(1) $$\sum_{j=1}^{n} u_j = u_{n+2} - u_2.$$

This formula can also be used to prove that $\lim \frac{u_{n-1}}{u_n}$ exists. Also

(2) $$u_{n+1}^2 = u_n u_{n+2} + (-1)^n \qquad \text{(prove by induction)}$$

The Pascal triangle connection.

								1		1	1	2				
							1		1		3	5				
						1		2		1		8	13			
					1		3		3		1		21			
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
	1		7		21		35		35		21		7		1	

Fibonacci Sequence in Pascal's Triangle

Beginning with each one (**1**) going down the left diagonal, sum up the diagonal entries where the diagonal slope is 1/3 (i.e. 3 cells right, 1 cell up, ...). This scheme generates the Fibonacci sequence.

The modern, general form: Given a, b, c, and d. Let

$$x_0 = a \qquad x_1 = b$$
$$x_{n+2} = cx_{n+1} + dx_n.$$

There are many results about such sequences, some similar to those already shown.

A cubic equation. In what appears at an attempt toward proving that solutions of cubic equations may not be constructible numbers, Fibonacci showed that the solution to the cubic equation

$$x^3 + 2x^2 + 10x = 20$$

can have no solution of the form $a + \sqrt{b}$, where a and b are rational. He gives an approximation 1; 22, 7, 42, 33, 4, 40 – best to that time, and for another 300 years. Note the use of sexagesimal numbers.

3.2 *Liber abbaci*

As a summary we may note that for the *Liber abbaci*:

- **Sources** – Islamic texts

- **Contents**
  - Rules for positional arithmetic,
  - Rules for the calculation of profits, currency, conversions, measurement
- **Problem types** – mixture problems, motion problems, container problems, Chinese remainder problem, quadratics, summing series
- **Methods** – wide and varied – most are original

Another Example. (A) If you give me a coin, we have the same.

(B) If I give you a coin, you have ten times what I have.

$$\begin{aligned} x &= \text{me} \\ y &= \text{you} \\ z &= x + y \end{aligned}$$

is the total amount.

Solution. Add $x + 1$ to both sides of the first equation to get

$$\begin{aligned} x + 1 &= \tfrac{1}{2}z \\ y + 1 &= \tfrac{10}{11}z. \end{aligned}$$

So $\underbrace{x + y}_{z} + 2 = (\tfrac{1}{2} + \tfrac{10}{11})z = \tfrac{31}{22}z$

$$\begin{aligned} \frac{9}{22}z &= 2 \\ z &= 44/9 \\ x &= \frac{44}{18} - 1 \quad y = \frac{32}{9} \\ &= \frac{26}{18} = \frac{13}{9}. \end{aligned}$$

3.3 *Liber quadratorum*

He also wrote *Liber quadratorum*, a brilliant work on intermediate analysis. This work was clearly a summary of number theory of the time. It was extensively quoted by Luca Pacioli in his book *Summa de arithmetica, geometrica proportioni et proportionalita* published in

1494 more than two centuries after the publication date. The book was also known to Tartaglia half a century later. Yet no manuscript was unavailable generally until the mid-nineteenth century, when a manuscript was uncovered by the medieval scholar Baldassarre Boncompagni in the Ambrosian Library in Milan. Boncompagni, in fact, corrected the edition he discovered and republished it in Latin.

Consider the Diophantine-like problem posed by **Master John of Palermo**. The number 5, added or subtracted from the square, the result will be the square of a rational. In modern form

$$r^2 + 5 = s^2 \qquad r^2 - 5 = t^2 \qquad \begin{array}{c} r, s, t \\ rational \end{array}$$

The solution of this problem appears as Proposition 17 of the 24 propositions in the work. Fibonacci's resolution is remarkably sophisticated. First, he defines the notion of **congruous numbers**: numbers of the form $ab(a+b)(a-b)$ if $a+b$ is even or $4ab(a+b)(a-b)$ if $a+b$ is odd. Such numbers he shows must be divisible by 24. Moreover, the system $x^2 + m = s^2$ and $x^2 - m = t^2$ has integers solutions only if m is congruous. Next, he shows that 5 is not congruous, but $12^2 \cdot 5$ is congruous. From this he is able to find a rational solution. Answer:$3\frac{5}{12}$.

3.3.1 Congruous numbers

Let a and b be integers. We say that the following numbers are *congruous*

$$\begin{array}{lll} ab(a+b)(a-b) & \text{if} \quad a+b & \text{is even} \\ 4ab(a+b)(a-b) & \text{if} \quad a+b & \text{is odd} \end{array}$$

Proposition. Congruous numbers are divisible by twelve.

Proof. We assume that $a+b$ is even. Thus either both a and b are even or both odd. In both even case we may write this as $a = 0 \pmod 2$ and $b = 0 \pmod 2$. Thus $a = 0 \pmod 2$ and a$b = 0 \pmod 2$, $a+b = 0 \pmod 2$ and $a-b = 0 \pmod 2$. Hence the expression $ab(a+b)(a-b)$ is divisible by 8. To show it is divisible by 3, we suppose several cases.

1. $a = 0 \pmod 3$. In this case the result holds.
2. $a = 1 \pmod 3$ and $b = 2 \pmod 3$. In this case $3| \, (a+b)$

3. $a = 1 \pmod 3$ and $b = 1 \pmod 3$ In this case $3 | (a - b)$
4. $a = 2 \pmod 3$ and $b = 2 \pmod 3$ In this case $3 | (a - b)$

The other cases follow by symmetry. In the case that $a + b$ is odd, this means that either a or b is is even and the other is odd. Thus we have the factor 8. To obtain the factor of 3 we can apply the same argument above. It remains to find a factor of 2. To this end since $a + b$ is odd, this means that even is even and the other is odd.

3.3.2 More results

In *Liber quadratorum* Fibonacci makes frequent use of the identities

$$\begin{aligned}(a^2 + b^2)(c^2 + d^2) &= (ac + bd)^2 + (bc - ad)^2 \\ &= (ad + bc)^2 + (ac - bd)^2,\end{aligned}$$

which had also appeared in Diophantus. Most of the results have a distinct number-theoretic flavor. The twenty-four propositions, all involving squares of numbers often in combination with other squares, are interesting and clever. As a further sampling we have:

Proposition 1. Find two square numbers which sum to a square number.
A rather clever proof is given based on the result $\sum_{j=1}^{n}(2j - 1) = n^2$, which is in fact proved as Proposition 4. Fibonacci adheres to the tradition that quantities should be measurable lengths or areas.

Proposition 2. Any square number exceeds the square immediately before it by the sum of the roots.

Proposition 3. Find two square numbers which sum to a square number.
A second proof of Proposition 1 is given.

Proposition 5. Find a square number that can be written as the sum of two squares in two different ways.
Typically a geometric proof is given.

The last and second last results extend far beyond any practical needs.
Proposition 23. Find three squares so that the sum of the first and

second as well as all three numbers are square numbers.

An English translation made from the Latin edition (Boncompagni, Baldassarre. *Liber guadratorum*. In Scritti di Leonardo Pisano. Rome, 1862.) by L. E. Sigler and was published in 1987. (L. E. Sigler,*The Book of Squares*, Academic Press, ISBN 0-12-643130-2)

3.4 Fibonacci on geometry

Fibonacci also wrote on geometry. In his *De practica geometriae*, 1220, he draws heavily upon the Greek masters, Euclid, Archimedes, and Ptolemy to name three. It was a practical treatise that included instructions on surveying. It also included instructions on how to find areas of segments and sectors of circles. To do this he needed a table of arcs and chords. His chord table is based on a circle of radius 21. (Note with the approximation of $\pi \approx 3\frac{1}{7}$, this gives the semicircle to be integral.) He uses Pisan measures of feet (6 to the rods) and unicae (18 to the foot) and points (20 to the unicae). He also shows how to interpolate the table, thus allowing the reader to compute areas for circles of other radii. Interestingly, when he gives discussion on calculating heights he does not use trigonometry, but rather uses similarity of triangles by using a pole of a given height and then using an angular siting together with marking off distances.

4 Medieval Universities

The modern university evolved from medieval schools known as *studia generalia*, recognized places of study open to students from all of Europe. As indicated earlier, these *studia* were created from the need to educate clerks and monks, at a level beyond the monastic schools. They included scholars from other countries, and this constituted a primary difference between the *studia* and the schools from which they grew.

The very earliest Western institution that could be termed a university was a 9th century medical school at Salerno, Italy. Drawing students from all of Europe, it was renown as a medical school. The first true universities, comprised of many disciplines, were founded at

Bologna late in the 11^{th} century, the University of Paris, founded between 1150 and 1170, and the University of Oxford in England, which was well established by the end of the 12^{th} century. The later two were composed of colleges, which in fact were endowed residence halls for scholars.

These universities were societies or guilds enjoyed wide ranging independence, given at the discretion of kings, emperors, and popes, who only required that neither heresy or atheism could be taught. The price of the independence was that they pay their own way. This required that the scholars charged tuition to gain a livlihood. As such they needed to satisfy the students on whom they depended for fees. As a consequence various universities were in vogue or not as hosts of students might migrate from one institution to another. Indeed, the University of Cambridge was founded by disgruntled students from Oxford.

From the 13^{th} century onwards, all major cities had a university. The curriculum consisted of the classical *trivium* and *quadrivium* of the classical age of Greece. Thus the subjects studied were logic - grammar - rhetoric - arithmetic - geometry - music - astronomy. Study focused on the works of the great philosophers such as Aristotle and Plato. Mathematical studies included the texts by Euclid and Nicomachus.

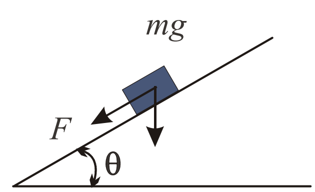
The first university, modern secular sense, was founded in Halle[2] (Germany) by Lutherans in 1694. This school renounced religious orthodoxy of all kinds, favoring rational and objective intellectual inquiry. Also, lectures were given in the vernacular (German) instead of Latin. Halle's innovations were later adopted by the University of Göttingen (founded in 1737) and subsequently by most universities.

5 Jordanus Nemorarius

(fl. 1220) Jordanus (fl. 1220) was younger than Fibonacci and was the founder of the Medieval school of **mechanics**. Almost nothing about him is known, except it is believed he taught at the new university in Paris about 1220. He wrote on geometry, arithmetic and mechanics. His fame was assured by his solution of a problem that eluded Archimedes, namely the problem of the inclined plane,

$$F = mg\sin\theta.$$

[2] The great set theorist, Georg Cantor, was a faculty member at Halle.



Inclined Plane

He wrote the book *Arithmetica*. In it he demonstrates a mastery of the theory of proportion as well as facility with quadratic terms. Some examples of his results:

Theorem. A multiple of a perfect[3] number is number is abundant. A divisor of a perfect number is deficient.

Note the **rhetorical base** significant for the use of letters instead of numerals for numbers. The uniqueness of the solution of division problems is considered by Jordanus as follows:

Theorem. If a given number is divided in two parts whose difference is given, each of the parts is determined.

Theorem. If a given number is divided into two parts whose product is determined each of the parts is determined.

Theorem. If a given number is divided into however many parts, whose continued proportions are given, then each of the parts is determined.

6 Nicole Oresme

Nicole Oresme (1323 - 1382), after studying theology in Paris, became bursar in the University of Paris and later dean of Rouen. In 1370 he was appointed chaplain to King Charles V as his financial advisor.

Oresme invented coordinate geometry before Descartes whereby he established the logical equivalence between tabulated values and their

[3]Recall that a number is termed *perfect* if its divisors add up to itself. A number is *abundant* if its divisors sum to a number greater than itself. A number is *deficient* if its divisors sum to a number smaller than itself. For example, 6 and 28 are perfect; 12 and 36 are abundant; 7 and 21 are deficient. In general, all prime numbers are deficient.

graphs. He proposed the use of a graph for plotting a variable magnitude whose value depends on another. It is possible that Descartes was influenced by Oresme's work since it was reprinted several times over 100 years after its first publication. Oresme also worked on infinite series which we shall discuss presently.

Another work by Oresme contains the first use of a fractional exponent, although, of course, not in modern notation. Oresme also opposed the theory of a stationary Earth as proposed by Aristotle and taught motion of the Earth — 200 years before Copernicus.

Latitude of forms. in about 1361 he conceived of the idea to visualize or picture the way things vary (function representation at an early stage). Everything measurable, Oresme wrote, is imaginable in the manner of continuous quantity. In this way he drew a velocity-time graph for a body moving with uniform acceleration. In this connection he used the terms latitude and longitude as we use abscissa and ordinate. His graphical representation is akin to our analytic geometry. His use of coordinates was not new however. (Apollonius) His main interest was in quadratures, and therefore he missed noticing functions and functional ideas *per se*.

The graphical representation of function, known as the *latitude of forms* was a popular topic from the time of Oresme to Galileo. His *Tractatus de figuratione potentiarum et mensurarum* was printed four times between 1482 and 1515. Oresme even suggested a three dimensional version of his latitude of forms.

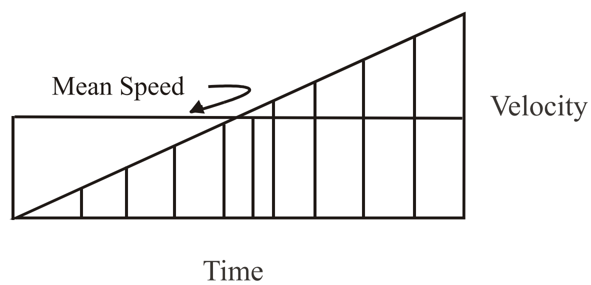
Oresme generalized Bradwardinés rule of proportion to include fractional powers giving the equivalents of our laws of exponents

$$x^m x^n = x^{m+n} \quad (x^m)^n = x^{mn}$$

suggested the use of irrational powers $\rightarrow x^{\sqrt{2}}$ but failed due to terminology and notation and ...

6.1 The Merton school

The Merton School was one of the colleges at Oxford where a step toward a more modern physics was advanced by Thomas Bradwardine. One of his main interests was the investigation of infinite decomposability of the continuum. He also considered geometrical shapes in terms of the points that comprise them. One problem that he exposed dated from Euclid's time, that being the angle between a curve and its tangent. He argued that if the angle is positive there results a contradiction, while if it is zero there can be no angle. A paradox? One of his greatest efforts was toward proving the **Mean Speed Rule** – distance traveled by an object in uniform acceleration, namely that the mean speed achieved halfway through the accelerated motion. Graphically, we have



$$\begin{aligned}\text{Distance} &= \text{sum of velocities} \times \text{time step} \\ &= \text{area}\end{aligned}$$

He goes further. Consider this:

Halves Area 1st half : Area 2nd half = 1 : 3 (Mean Speed Rule)

Thirds: $A_{1st} : A_{2nd} : A_{3rd} = 1 : 3 : 5$

Fourths: $A_{1st} : A_{2nd} : A_{3rd} : A_{4th} = 1 : 3 : 5 : 7$

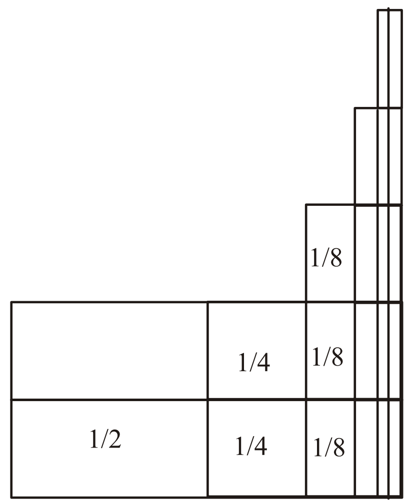
and so on. In as much as the sums of the odd integers are n^2, the total distance covered varies as the square of time $\star$ Galileo: law of motion.

7 Infinite series

In the fourteenth century mathematicians had imagination and precision of thought, but lacked algebraic and geometric facility. Hence, they could at most equal the ancients in the same area. However, as we have seen they ventured into new areas. Another direction was toward infinite series. We emphasize here again the importance of a new philosophical viewpoint which permits new thoughts unconfined by past taboos. Nicole Oresme ventured boldly in this new direction without the *horror infiniti* of the Greeks. Among the many series he summed was:

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{n}{2^n}+\cdots=2$$

He produced a beautiful geometric proof.



Rearrangement of areas? or
Summing an infinite series?

This appearance of computing infinite sums may be illusory if taken from Oresme's viewpoint. Recall that at the time, philosophy was still very much Aristotelian, and Aristotle's conception of infinity was essentially **temporal**. An infinite process is one which does not end. But could this be applied to permanent objects? There were tensions.

Aristotle views that continuous objects were infinitely divisible. But when one divides a length into halves, then one of the halves into halves, one of those quarters into halves and so on indefinitely. Are those pieces really there? Aristotle would say they are only **potentially** there.

Mathematicians such as William of Ockham and Gregory of Rimini maintained that they were indeed there —- but there was no last member. This is the usual problem in dealing with the infinite.

An interpretation. What Oresme may have been doing is experimenting with moving these parts around. He conceived of two squares, each a foot in length. He divided the second square into proportional parts by means of vertical slices. These parts were then moved one on top of the other, creating a vertical tower. The total area was two square feet. Oresme's main interest, however, was in showing how a finite object could be infinite in this respect.

Robert Suiseth, (or Swineshead) (fl. ca. 1350), an English logician, better known as **Calculator** solved the following infinite series problem :

> If throughout the first half of a given time interval a variation continues at a certain intensity, throughout the next quarter of the interval at double this intensity, throughout the following eighth at triple the intensity and so ad infinitum; then the average intensity for the whole interval will be the intensity of the variation during the second subinterval.

This is equivalent to saying the sum of the series

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots=2.$$

Calculator gave a long and tedious proof, not knowing the graphical representation.

Alternate proof: (modern)

$$\begin{aligned}
\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots \\
&= \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots \\
&\quad +\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots \\
&\quad\quad +\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots \\
&\quad\quad +\cdots \\
&= 1+\frac{1}{2}(1)+\frac{1}{4}(1)+\cdots \\
&= 2
\end{aligned}$$

⋆ Oresme also summed

$$\frac{1\cdot 3}{4}+\frac{2\cdot 3}{16}+\frac{2\cdot 3}{64}+\cdots+\frac{2\cdot 3}{4^n}+\cdots \quad =2.$$

⋆ He proved the harmonic series diverges by grouping

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots+\frac{1}{n}+\cdots$$

Recall, the standard grouping argument to establish the divergence of the harmonic series. We first group $\frac{1}{2}$ with itself. Then group $\frac{1}{3} + \frac{1}{4}$. This quantity is greater than $\frac{1}{2}$. Next group $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$. Again this quantity is greater than $\frac{1}{2}$. Continue in this fashion taking the next eight terms, then the following sixteen terms, and so on. Each time we obtain a lower bound of $\frac{1}{2}$. We thus establish that partial sums of this series can be made greater than any multiple of $\frac{1}{2}$. This implies that the harmonic series diverges.

8 The conflict between religion and philosophy

A conflict was gradually brewing within the social structure of priests and philosophers. On the one hand the clergy was instructed in the required Peripatetic physics, and on the other hand was the dogma of Christian, Mohammedan, and Jewish belief in eternal souls. Of all subjects, the conflict centered on the arcane subject of *infinity*. The repercussions of this conflict was not an open dismissal of one set of beliefs in favor of the other, but a seeding of new ideas that would bear fruit in the next several centuries.

Let us review that for Aristotle the entire theory of infinity is encapsulated in four statements.

1. The existence of an actual infinity of distinct objects is contradictory.
2. A multitude of objects may be infinite potentially, and a finite number of objects can be made larger by addition.
3. The existence of an actual continuous infinite magnitude is contradictory.
4. The existence of a potential continuous infinity in magnitude is an impossibility. Essentially, magnitudes cannot become indefinitely large because the "container" is finite. That is the world is limited.

Now Aristotle could not do without potential infinity because of the self-evident truths all of which imply indefinite processes:

1. Number can be indefinitely augmented by addition.

2. Continuous magnitude can be indefinitely subdivided.
3. Time had no beginning and will have no end.

Also, Aristotle made no special provision for the beginning of mankind. In his philosophy, mankind always existed. From this, and being mindful that religious beliefs are among the most powerful we have, it is evident that a conflict with a simple deduction: If time has no beginning and life has no beginning, then there must be an actual infinitude of souls (of the dead).

We may ask how Aristotle, or Peripatetic philosophy, might handle this problem. He might deny the individual survival of the soul, that it dies with the body, as was the opinion of Alexander of Aphrodisias. Or as the Spanish-Arabic philosopher Averroes[4] suggested on his behalf, all souls unite into a single intelligence common to all humanity.

The pagan philosophers would not be worried by the objection, and the neo-Platonists believe souls were limited in number via the process of reincarnation. The Stoics denied any belief in the immortality of the soul. Hence, the Greek philosophy was not challenged to resolve this issue. However, all three religions, Christianity, Judaism, and Islam, did believe in the individual survival of the souls and moreover denied reincarnation. However, if the past was finite there could be but a finite number of souls. Thus the problem is shifted to reconcile

- The Peripatetic metaphysical doctrine that neither the world nor humanity had a beginning, and
- The religious belief that the human souls subsists after death, distinct from other souls, and exempt from reincarnation.

The Islamic philosopher Avicenna[5], allowed that the impossibility

[4] Abul Walid Mahommed Ibn Achmed, Ibn Mahommed Ibn Roschd (Averroes), 1128 - 1198, was born in Cordova, the son and grandson of judges. He devoted himself to jurisprudence, medicine, and mathematics, as well as to philosophy and theology, and particularly the philosophy of Aristotle. Though his commentaries were based on a rather imperfect Arabic translation of the Syriac version of the Greek text, they were of great influence in determining the philosophical and scientific interpretation of Aristotle in the Christian, Jewish and Islamic worlds. His influence waned somewhat during the renaissance though St. Thomas Aquinas used the commentaries of Averroes as his model.

[5] ABN ALI AL HOSAIN IBN ABDALLAH IBN SINA, called by the Latins AVICENNA, (987-1037) was an Iranian physician, and probably the most famous and influential of the philosopher-scientists of Islam. He was particularly noted for his contributions in the fields of Aristotelian philosophy and medicine. He composed the Book of Healing , an expansive philosophical and scientific encyclopedia. This book was translated into Latin in the 12th century and strongly influenced scholastic thinking.

of an actual infinite multitude is *not* absolute. There is, he maintained, a basic difference between
(i) An infinite multitude of objects occupying a position, and
(ii) An infinite multitude of objects stripped of all position. This argument seems to set a criterion to differentiate which infinities can be actual and which potential, but there are exceptions. Avicenna continues to argue that each thing has a cause and each cause has a cause and sone. But there can be no infinite hierarchy of causes. There must be a first cause. As causes do not occupy position, this exception proves difficult to maintain or reason without the axiom that it can not be. So, with Avicenna, we have that some actual infinities are possible, but some are not. He distinguished between things whose nature implies a natural or determined order and between things having no particular order. The former cannot be actually infinite, while the latter can. There are in the end four infinities, of which two exist and two do not exist.

1. The movement of heaven has neither beginning nor end.
2. There is an infinity of human souls, distinct and separated from their bodies.
3. The may not be a body infinite in extent.
4. There can be no infinite series of causes for which no first cause can be obtained.

All of this was rather unsettling to al-Gazali, who when he returned to the teaching of the Koran denied the existence of actual infinity altogether. Adding to his collection of issues was the self-contradictory nature of two particular actual infinities: if the world has no beginning, then both the sun and Saturn have both made an infinity of revolutions, but the ration of these is a determined number.

To avoid the mounting problems with the infinity of souls, the philosophers took the position that the world did have a beginning. This is the current dogma of all three religions and maintained by Faith alone. This was the dogma that Saint Thomas Aquinas defended ardently. Thus the number of souls is finite. The rest of the infinities consisting of planetary orbits, sequences of causes all disappeared as well. Thus the conflict was resolved. However, the concept was loosed upon all thinkers. Most specifically, with the experiments with various actual infinities philosophers and mathematicians have a reduced apprehension about the infinity concept. Indeed, by simply allowing the (a) world with no beginning, the infinities come right back. This seems to be a most interesting interplay of philosophy and religion that ultimately benefitted mathematics.

9 Decline of Medieval learning

- Black Plague
- Hundred Years War (England-France)
- War of the Roses
- Shift from England and France to
  - Italy
  - Poland
  - Germany

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ierre Duhem, *Medieval Cosmology, Theories of Infinity, Place, Time, Void, and the Plurality of Worlds*, Chicago University Press, Chicago, 1985.