Problem Set Ten: The Ascoli Theorem

Definitions: (a) A set $M$ of real valued functions on $S$ is pointwise bounded iff 
\[ \forall s \in S \exists b(s) > 0 \forall f \in M, |f(s)| \leq b(s). \]
(b) A set $M$ of real valued functions on $S$ is uniformly bounded iff it is bounded in the metric space $B(S)$, equivalently, \[ \exists b > 0 \forall f \in M \forall s \in S, |f(s)| \leq b. \]
(c) A set $M$ of continuous functions on $S$ is equicontinuous iff 
\[ \forall \varepsilon > 0 \exists \delta > 0 \forall f \in M, d(s, t) < \delta \Rightarrow |f(s) - f(t)| < \varepsilon. \]
This condition amounts to the functions in $M$ being "uniformly uniformly continuous."

Examples: Let $\varphi(s) = \sin(s), 0 \leq s \leq \pi$, and $\varphi(s) = 0, \pi \leq s$.
(a) The functions $f_n(s) = n\varphi(ns)$ converge pointwise to zero and are pointwise bounded on $S = [0, \pi]$, but neither uniformly bounded nor equicontinuous on $S = [0, \pi]$.
(b) The functions $g_n(s) = \varphi(ns)$ are uniformly bounded but not equicontinuous on $S = [0, \pi]$.
(c) The functions $h_n(s) = n$ are equicontinuous but not uniformly bounded on $S = [0, \pi]$.

Lemma: If $S$ is any set, $D \subset S$ is a countable subset and $(f_n)$ is pointwise bounded on $S$, then $(f_n)$ has a subsequence that converges pointwise on $D$.

Example: Let $Z$ denote the integers and $D = \{k/10^n : n \geq 1 \text{ and } 0 \leq k \leq 10^n\}$. The functions 
\[ f_n(x) = d(10^n x, Z) \]
converge pointwise to zero on $D$ but don't converge pointwise on $[0,1]$.

Theorem (Ascoli): Let $S$ be compact metric. Any sequence in $C(S)$ that is pointwise bounded and equicontinuous has a uniformly convergent subsequence.

Corollary: Let $S$ be compact metric. If $(f_n)$ is equicontinuous and converges pointwise to $f$, then $f$ is continuous and $(f_n)$ converges uniformly to $f$.

Theorem (Ascoli): Let $S$ be compact metric. A set $M \subset C(S)$ is compact in the uniform norm iff $M$ is closed, pointwise bounded and equicontinuous.

Example: The set of functions $f$ in $C[0,1]$ with $f(0) = 0$ and $\text{Lip}(f) \leq 1$ is closed, uniformly bounded and equicontinuous.

PROBLEMS

Problem 10-1: Bounds on derivatives often imply Lipschitz estimates and equicontinuity. Show that each of the following sets is equicontinuous on $[0,1]$:
(a) the set of all continuously differentiable functions on $[0,1]$ with $\|f'\|_u \leq 1$;
(b) the set of all continuously differentiable functions on $[0,1]$ with $\|f'\|_2 \leq 1$. 
Problem 10-2 : Let \( \{f_n\} \) be a sequence \( C[a,b] \) which is Cauchy in 2-norm. Prove that if \( \{f_n\} \) is

equi-continuous then \( \{f_n\} \) is Cauchy in the uniform norm.

Notation: The translate of \( f \) by \( t \), denoted by \( f_t \), is the function \( f_t(x) = f(x + t) \), which is defined

wherever \( x + t \in \text{domain}(f) \).

Problem 10-3: If \( f : [0, \infty) \to \mathbb{R} \) is continuous and \( \lim_{x \to \infty} f(x) = L \) exists, then \( f \) is bounded and

uniformly continuous.

Problem 10-4: Let \( f : [0, \infty) \to \mathbb{R} \) be continuous with \( \lim_{x \to \infty} f(x) = L \). Define \( T : [0, \infty) \to B[0, \infty) \) so

that \( T(t) \) is the translate of \( f \) by \( t \), i.e., by \( T(t)(x) = f(x + t) \). Prove the following.

(a) \( T : [0, \infty) \to B[0, \infty) \) is bounded, uniformly continuous and \( \lim_{t \to \infty} \|T(t) - L\|_u = 0 \).

(b) \( \{f_t : t \geq 0\} \) is bounded and totally bounded in \( B(0, \infty) \).

(c) \( \{f_t : t \geq 0\} \cup \{L\} \) is compact in \( B(0, \infty) \); here \( L \) denotes the constant function \( g(x) = L \).

(d) For \( f(x) = \exp(-|x|) \) the set \( \{f_t : t \geq 0\} \) is totally bounded in \( B(0, \infty) \), but \( \{f_t : -\infty < t < \infty\} \) is not

totally bounded in \( B(-\infty, \infty) \).

Problem 10-5: If \( f : \mathbb{R} \to \mathbb{R} \) is continuous and has period \( p > 0 \), then the set of restrictions

\( \{f_t \}_{t \in [0, p]} : -\infty < t < \infty \) is compact in \( C[0, p] \).

Problem 10-6: Let \( K \) be a continuous function on \([a, b] \times [c, d] \).

(a) For \( f \in C[c, d] \) the function \( F(s) = \int_c^d f(t) K(s, t) \, dt \) is in \( C[a,b] \).

(b) Define \( T : C[c, d] \to C[a, b] \) by \( (Tf)(s) = \int_c^d f(t) K(s, t) \, dt \). Check that \( T \) is a linear transformation,

i.e., for all \( f, g \in C[c, d] \) and scalars \( c \), \( T(f + g) = T(f) + T(g) \) and \( T(cf) = c \cdot T(f) \).

(c) If \( B = \sup \{ \int_c^d |K(s, t)| \, dt : c \leq s \leq d \} \) then \( \|T(f) - T(g)\| \leq B \|f - g\| \) for all \( f, g \in C[c, d] \).

(d) The set \( \{T(f) : \|f\| \leq 1\} \) is uniformly bounded and equicontinuous in \( C[a,b] \).

(e) Is there an \( r > 0 \) so that \( \{g \in C[a, b] : \|g\| \leq r\} \subset \{T(f) : \|f\| \leq 1\} \) ?