FACT FROM ADVANCED CALCULUS (Intermediate Value Theorem): If \( f : [a, b] \to \mathbb{R} \) is continuous and \( c \) is any number between \( f(a) \) and \( f(b) \), then \( c = f(x) \) for some \( x \in [a, b] \).

Definition: A **fixed point** of a function \( f : S \to S \) is a point \( p \in S \) for which \( f(p) = p \). A metric space \( S \) has the **fixed point property** (FPP) iff every continuous \( f : S \to S \) has at least one fixed point.

Theorem: Any interval \([a, b]\) with the usual metric has the fixed point property.

Example: The circle \( S = \{ x \in \mathbb{R}^2 : \| x \|_2 = 1 \} \) does not have the fixed point property. For instance the rotation \( f : S \to S \) given by \( f(s, t) = (-t, s) \) has no fixed point. Does the Cantor set have FPP?

An straightforward way to attempt to find fixed points is by iteration.

Theorem: Let \( S \) be a metric space and \( f : S \to S \) be continuous. For a given \( x_0 \in S \) define \( (x_n) \) recursively by \( x_{n+1} = f(x_n), n \geq 0 \). If \( x_n \to p \) then \( p \) is a fixed point of \( f \).

Examples: Each function maps \([0, 1]\) into itself.
(a) For \( f(x) = (1 + x) / 3 \) the fixed point is \( x = 1/2 \) and \( |f(x) - 1/2| = (1/3)|x - 1/2| \) for all \( x \). The iterates converge to \( 1/2 \) for any \( x_0 \in [0, 1] \).
(b) For \( f(x) = 1 - x \) the fixed point is \( x = 1/2 \) and \( |f(x) - 1/2| = |x - 1/2| \) for all \( x \). The iterates converge if \( x_0 = 1/2 \).
(c) For \( f(x) = \max\{0, 2x - 1\} \) the fixed points are \( x = 0 \) and \( 1 \). Iterates decrease to zero unless \( x_0 = 1 \).

In the last examples convergence is determined by the size of the ratios \( |f(x) - f(p)| / |x - p| \).

Theorem: Let \( S \) be a compact metric space. If \( f : S \to S \) is a function with \( d(f(x), f(z)) < d(x, z) \) whenever \( x \neq z \), then \( f \) has a unique fixed point.

Example: The last result can fail without compactness. \( f(x) = x + x^{-1} \) defined a function \( f : [2, \infty) \to [2, \infty) \) with no fixed point but with \( \left| f(x) - f(z) \right| < \left| x - z \right| \) whenever \( x \neq z \).

**Contraction Mapping Theorem** (or Banach Fixed Point Theorem): Let \( S \) be a complete metric space and \( f : S \to S \) be a function with Lipschitz constant \( \alpha = \text{Lip}(f) < 1 \).
(a) \( f \) has a unique fixed point \( p \).
(b) For any \( x_0 \in S \) the sequence \( (x_n) \) defined by \( x_{n+1} = f(x_n), n \geq 0 \), converges to \( p \), and \( d(p, x_n) \leq \frac{\alpha^n}{1 - \alpha} d(x_1, x_0) \) for all \( n \).
Examples: The last result can fail when \( \text{Lip}(f) = 1 \). \( f(s, t) = (-t, s) \) defines a function from the unit circle to itself with no fixed point but with \( \text{Lip}(f) = 1 \). The last result can fail if \( S \) is not complete. On \( S = [0,1) \), \( f(s) = (1 + s)/2 \) has Lipschitz constant \( 1/2 \) but no fixed point.

PROBLEMS

Problem 11-1. Let \( S \) be a metric space and \( f : S \to S \) be a function with a fixed point \( p \).

Each part is about a sequence of iterates defined by \( x_n+1 = f(x_n), \ n \geq 0 \).

1. Assume that \( \exists r > 0 \exists \alpha \in (0,1) \) such that \( d(x, p) < r \Rightarrow d(f(x), p) \leq \alpha d(x, p) \).

Prove, say by induction, that \( d(x_0, p) < r \Rightarrow d(x_n, p) \leq \alpha^n d(x_0, p) \) for all \( n \).

Conclude that \( (x_n) \) converges to \( p \).

2. Assume that \( \exists r > 0 \exists \beta \in (1, \infty) \) such that \( d(x, p) < r \Rightarrow d(f(x), p) \geq \beta d(x, p) \).

Prove that either

(i) \( \forall m \in N \exists n \geq m \) such that \( d(x_n, p) \geq r \), or

(ii) \( \exists m \in N, x_m = p \).

[Argue that not (i) implies (ii)]

Conclude that \( (x_n) \) cannot converge to \( p \) unless \( x_m = p \) for some \( m \).

Problem 11-2. Let \( f(x) = |2x - 1| \). Parts (c), (d) and (e) refer to sequences of iterates defined recursively by \( x_{n+1} = f(x_n), \ n \geq 0 \).

(a) Check that \( f \) is continuous and \( 0 \leq x \leq 1 \) implies \( 0 \leq f(x) \leq 1 \).

(b) Verify that \( f \) has fixed points \( 1/3 \) and \( 1 \) as a function \( f : [0,1] \to [0,1] \) .

(c) Find \( x_n \) for \( n = 1 \) through \( 6 \) for the sequence with \( x_0 = 2/3 \). Repeat for the sequences with \( x_0 = 1/5 \) and \( x_0 = 1/7 \) .

(d) Use part (2) of Problem 10-1 to show that if a sequence of iterates converges to a fixed point \( p \), then \( x_m = p \) for some \( m \).

(e) Prove that if \( x_0 \) is a rational number then \( \exists m \in N \exists p \in N, n \geq m \Rightarrow x_n+p = x_n \). Sequences with the last property are called eventually periodic.

Definitions: (a) An interval \( I \) means any sort of real interval or ray; \([a,b], [a,b), (a,b], (a,b), [a, \infty), (a, \infty), (-\infty, b], (-\infty, \infty) = R \). Let \( I \) be an interval, \( f : I \to R \) a function and \( p \) be in the interior of \( I \).

(b) \( f \) is differentiable at \( c \) iff \( \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \) exists. The limit is the derivative of \( f \) at \( c \) and is denoted by \( f'(c) \).

(c) Equivalently, \( f \) is differentiable at \( c \) with derivative \( f'(c) \) iff

\[ \forall \varepsilon > 0 \exists \delta > 0, 0 < |x - c| < \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon \]

(d) \( f \) is differentiable on \( I \) iff \( f \) is differentiable at each point \( c \) in \( I \).
Problem 11-3. Let I be an interval, let \( f : I \to I \) be a function with fixed point \( p \), and assume that \( f \) is differentiable at \( p \).
1. If \( |f'(p)| < 1 \) then \( f \) has the property stated in part (1) of Problem 10-1.
2. If \( |f'(p)| > 1 \) then \( f \) has the property stated in part (2) of Problem 10-1.

Problem 11-4. Let \( S \) be a compact metric space. You know that if \( f : S \to S \) is a function with \( d(f(x), f(z)) < d(x, z) \) whenever \( x \neq z \), then \( f \) has a unique fixed point \( p \). Prove that for any \( x_0 \in S \), the sequence defined recursively by \( x_{n+1} = f(x_n), n \geq 0 \), converges to \( p \).

Problem 11-5. Let \( \alpha < 1 \) and \( f : \mathbb{R} \to \mathbb{R} \) be a Lipschitz function with \( |f(x) - f(z)| \leq \alpha |x - z| \) for all \( x \) and \( z \). Prove that if \( f(0) > 0 \) then \( f \) has a fixed point in the interval \( (0, \frac{f(0)}{1 - \alpha}] \).

Problem 11-6. For a function \( f : S \to S \), define \( f^{[n]} : S \to S \) to be the composition of \( f \) with itself \( n \) times, i.e., by \( f^{[n]} = f \circ f \circ ... \circ f \). Prove that if \( \alpha = \text{Lip}(f^{[n]}) < 1 \) and \( p \) is the unique fixed point of \( f^{[n]} \), then \( p \) is also a fixed point of \( f \).