Problem Set Twelve: Applications of the Banach Fixed Point Theorem

MEAN VALUE THEOREM (Fact from Advanced Calculus): If \( g : [a, b] \to \mathbb{R} \) is continuous on \([a,b]\) and differentiable on \((a,b)\) then there is a \( w \in (a, b) \) so that \( g(b) - g(a) = g'(w)(b - a) \).

Special Case of the Implicit Function Theorem: Let \( S \) be a compact metric space, \( F : [a, b] \times S \to \mathbb{R} \) be continuous and \( (x_0, s_0) \in (a, b) \times S \) be a point at which \( F(x_0, s_0) = 0 \).

If \( \frac{\partial F}{\partial x} \) exists, is bounded and is bounded away from zero on \((a, b) \times S\) and then

1. \( \exists \delta > 0 \) \( \exists \varphi : B(s_0, \delta) \to (a, b) \) with \( \varphi \) continuous, \( \varphi(s_0) = x_0 \), and \( F(\varphi(s), s) = 0 \) whenever \( s \in B(s_0, \delta) \), and
2. any two such functions agree on their common domain.

The Method of Proof is to find a fixed point of \( (T \varphi)(s) = \varphi(s) - F(\varphi(s), s)/M \).

The other application is the Existence and Uniqueness Theorem for First Order Ordinary Differential Equations.

Theorem: Let \( D = [a - \delta_0, a + \delta_0] \times [b - \varepsilon_0, b + \varepsilon_0] \), \( f : D \to \mathbb{R} \) be continuous, and suppose there are constants \( M \) and \( L \) such that

1. \( \forall (x, y) \in D, |f(x, y)| \leq M \) and
2. \( \forall (x, y), (x, z) \in D, |f(x, y) - f(x, z)| \leq L |y - z| \).

If \( \delta \) is chosen to satisfy \( 0 < \delta \leq \delta_0, \delta M \leq \varepsilon_0 \) and \( \delta L < 1 \), then the initial value problem \( y'(x) = f(x, y(x)), y(a) = b \) has a unique solution defined at least on \([a - \delta, a + \delta]\).

Example: In the initial value problem \( y' = \sqrt{x + y}, y(4) = 5, f(x, y) = \sqrt{x + y} \) is continuous on the rectangle \( D = [3, 5] \times [4, 6] \).

1. \( \forall (x, y), (x, z) \in D, |f(x, y)| \leq \sqrt{11} \) and \( |f(x, y) - f(x, z)| \leq \frac{1}{2\sqrt{11}} |y - z| \).
2. The existence and uniqueness theorem guarantees a solution defined at least on \([3.7, 4.3] \subset [4 - 1/\sqrt{11}, 4 + 1/\sqrt{11}] \).

Method of Proof: The initial value problem and the integral equation \( y(x) = b + \int_a^x f(t, y(t)) \, dt \) have the same solutions. The solution to the integral equation is the fixed point of the function \( T \) defined by \( T(y)(x) = b + \int_a^x f(t, y(t)) \, dt \). The theorem’s hypothesis allows the Contraction Mapping Theorem to be applied with \( T \) defined on a certain closed subset \( X \) of \( C[a - \delta, a + \delta] \). The iterative sequence \( y_{n+1}(x) = b + \int_a^x f(t, y_n(t)) \, dt \) that converges to the solution is called the sequence of Picard iterates.

Example: For the initial value problem of the preceding example
(3) \( y_0(x) = 3x - 7 \) has the same value and derivative as the solution at \( a = 4 \).

(4) With \( y_0(x) = 3x - 7 \) the next Picard iterate is \( y_1(x) = \frac{1}{2} + \frac{1}{6} (4x - 7)^{3/2} \).

The assertions in the following proposition will also be used in proving the Peano Theorem.

**Proposition**: Let \( D = [a - \delta_0, a + \delta_0] \times [b - \varepsilon_0, b + \varepsilon_0] \) and \( f : D \to \mathbb{R} \) be a continuous function with \( |f(x, y)| \leq M \) for all \((x, y) \in D\). Assume \( 0 < \delta \leq \delta_0 \) and \( \delta M \leq \varepsilon_0 \).

If \( X = \{ y \in C[a - \delta, a + \delta] \} : y(a) = b \) and \( y \left( [a - \delta, a + \delta] \right) \subseteq [b - \varepsilon_0, b + \varepsilon_0] \) then \( X \) is convex and closed in \( C[a - \delta, a + \delta] \). Further for each \( y \in X \)

1. the function \( t \mapsto f(t, y(t)) \) is defined and continuous on \([a - \delta, a + \delta]\),
2. the function \( (Ty)(x) = b + \int_a^x f(t, y(t)) \, dt \) is in \( X \), and
3. \( (Ty)'(x) = f(x, y(x)) \) on \([a - \delta, a + \delta]\).

**PROBLEMS**

Problem 12-1. (a) Let \( r \) be a simple root of the polynomial \( P(x) = \sum_{k=0}^{n} a_k x^k \). Use the Implicit Function Theorem to discuss the effect on the root of small changes in the coefficients.

(b) \( F(3, (6, 9)) = 0 \) for \( F(x, (y, z)) = x^2 - y x + z \). Is there a positive delta and a continuous \( \varphi : B((6,9), \delta) \to \mathbb{R} \) with \( \varphi(6,9) = 3 \) and \( F(\varphi(x, (y, (x, y))) = 0 \) whenever \((x, y) \in B((6,9), \delta) \)? Why (not)?

Problem 12-2. Check that this follows from the Existence Uniqueness Theorem. Let \( U \subset \mathbb{R}^2 \) be an open set and \((a, b) \in U \). If \( f : U \to \mathbb{R} \) is continuous and \( \frac{\partial f}{\partial y} \) exists and is continuous on \( U \) then there is an open interval containing \( a \) on which the initial value problem \( y'(x) = f(x, y(x)) \), \( y(a) = b \), has a unique solution.

Problem 12-3. This problem is about the initial value problem \( y' = x + \frac{1}{y}, y(0) = 3 \).

(a) For \( f(x, y) = x + y^{-1} \) on \( D = [-1,1] \times [2,4] \), find constants \( M \) and \( L \) so that

\[ \forall (x, y), (x, z) \in D, \left| f(x, y) \right| \leq M \text{ and } \left| f(x, y) - f(x, z) \right| \leq L \left| y - z \right| . \]

(b) Find \( \delta > 0 \) so that the initial value problem has a unique solution on \([\delta, \delta] \).

(c) Find the linear function \( y_0(x) \) that has the same value and derivative as the solution at \( a = 0 \), and then find the next Picard iterate \( y_1(x) \).

Problem 12-4. This problem is about the initial value problem \( y' = x - y, y(0) = 2 \).

(a) Check that the solution is \( y = x - 1 + 3e^{-x} \).

(b) For \( y_0(x) = 2 - 2x \) show, say by induction, that for \( n > 0 \) the Picard iterates are

\[ y_n = x - 1 + 3 \sum_{k=0}^{n-1} \frac{(-x)^k}{k!} . \]