Problem Set Twelve: The Brouwer, Schauder and Peano Theorems

Definition: A function \( f : S \rightarrow T \) between metric spaces is a homeomorphism iff \( f \) is 1-1, onto, continuous and has a continuous inverse. Two spaces are homeomorphic \((S\sim T)\) iff there is a homeomorphism between them.

Theorem: If \( S \) is compact and \( f : S \rightarrow T \) is 1-1, onto, and continuous, then \( f \) is a homeomorphism.

Examples: The Cantor set \( C \) and \( \Delta \sim \Delta \) are homeomorphic; \( \Delta \sim \Delta \times \Delta \) and \( C \sim C \times C \); \([0,1]\) is not homeomorphic \([0,1]\times[0,1]\).

Example: In \( \mathbb{R}^n \) the closed unit balls in the Euclidean and box norms are homeomorphic. To write a homeomorphism \( f : CB(0,1) \rightarrow CB \) let \( f(0) = 0 \) and \( f(x) = \left\| x \right\|_{\infty}^{-1} x \) for \( x \neq 0 \).

Theorem: If \( S \) and \( T \) are homeomorphic and \( S \) has the fixed point property then so does \( T \).

Brouwer Fixed Point Theorem: For each \( n \geq 1 \) the closed unit ball in \( \mathbb{R}^n \) with Euclidean norm has the fixed point property.

Lemma: The standard simplex in \( \mathbb{R}^n \) is the set of \( n \)-tuples \( s = (t_1, t_2, \ldots, t_n) \) with each \( t_k \geq 0 \) and \( \sum_{k=1}^{n} t_k = 1 \). The standard simplex is homeomorphic to the closed unit ball in \( \mathbb{R}^{n-1} \).

Lemma (Partitions of Unity): If the open balls \( B(x_k, \varepsilon_k) \), \( 1 \leq k \leq m \), cover the metric space \( S \), then there are \( m \) continuous functions \( f_k : S \rightarrow [0,1] \) so that

(i) for each \( k \) the function \( f_k = 0 \) on the complement of \( B(x_k, \varepsilon_k) \), and

(ii) \( \sum_{k=1}^{m} f_k(x) = 1 \) for all \( x \) in \( S \).

Lemma: If \( K \) is a convex subset of a vector space, \( x_1, x_2, \ldots, x_n \in K \), and \( s = (t_1, t_2, \ldots, t_n) \) is in the standard simplex, then \( \sum_{k=1}^{n} t_k x_k \in K \).

Schauder Fixed Point Theorem: Let \( E \) be a Banach space and \( K \subset E \) be a closed and convex. If \( T : K \rightarrow K \) is continuous and \( T(K) \) is totally bounded, then \( T \) has a fixed point.

Corollary: Any compact, convex set in a Banach space has the fixed point property.

Peano Theorem: Let \( U \subset \mathbb{R}^2 \) be an open set and \((a, b) \in U\). If \( f : U \rightarrow \mathbb{R} \) is continuous then there is an open interval containing \( a \) on which the initial value problem \( y'(x) = f(x, y(x)) \), \( y(a) = b \), has a solution.

Example: Uniqueness of solutions may fail. The initial value problem \( y' = 2^x \), \( y(0) = 0 \), has solutions \( y_1(x) = 0 \) and \( y_2(x) = x^2 \text{sign}(x) \).
Definitions: Let $X$ be a metric space and $Y$ be a subset of $X$. A continuous function $r : X \to Y$ is a retraction of $X$ onto $Y$ iff $r$ is onto and $r(y) = y$ for all $y$ in $Y$. $Y$ is a retract of $X$ iff there is a retraction from $X$ onto $Y$. For example each interval $[a,b]$ is a retract of the reals $\mathbb{R}$ and $r(x) = \max\{a, \min\{x, b\}\}$ is a retraction from $\mathbb{R}$ onto $[a,b]$.

Problem 12-1: (a) If $Y$ is a retract of $X$ and $X$ has the fixed point property, then so does $Y$.
(b) $\{x \in \mathbb{R}^n : \|x\| = 1\}$ is not a retract of the unit ball $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$. (Hint; use Brouwer Theorem.)

Problem 12-2: In each case prove there is a retraction $r : X \to Y$ with the indicated Lipschitz norm:
(a) $X = C(S)$, $Y = CB(f, \varepsilon)$ any closed ball in $C(S)$, $\text{Lip}(r) \leq 1$;
(b) $X = \mathbb{R}^n$ with Euclidean norm, $Y = CB(x, \varepsilon)$ any closed ball in $\mathbb{R}^n$, $\text{Lip}(r) \leq 1$;
(c) $X$ any normed space, $Y$ any closed ball in $X$, $\text{Lip}(r) \leq 3$.

Problem 12-3: Prove the indicated facts.
(a) The Cantor set does not have the fixed point property.
(b) The Hilbert Cube has the fixed point property. (Hint; use Brouwer Theorem)
(c) The closed unit ball $CB(0,1)$ in $C(S)$ does not have the fixed point property.