Problem Set Thirteen; The Baire Category Theorem

Definitions: Let $(S,d)$ be a metric space. Recall a set $U \subseteq S$ is open iff for each $x \in U$ there is an open ball $B(x,r)$ so that $B(x,r) \subseteq U$. A subset $F$ of $S$ is closed iff its complement $F^c = S \setminus F$ is an open set.

Theorem: (1) The union of any collection of open sets is open.
(2) The intersection of any finite number of open sets is open.
(3) If $f : S \to T$ is continuous and $U \subseteq T$ is open then $f^{-1}(U) = \{ x \in S : f(x) \in U \}$ is open in $S$.

Corollary: (1) The intersection of any collection of closed sets is closed.
(2) The union of any finite number of closed sets is closed.
(3) If $f : S \to T$ is continuous and $F \subseteq T$ is closed then $f^{-1}(F) = \{ x \in S : f(x) \in F \}$ is closed in $S$.

Definition: The diameter of a bounded set $M$ in a metric space is $\text{diam}(M) = \sup \{ d(x,y) : x \& y \in M \}$.

Notice that a closed ball $CB(x,r)$ has diameter at most $2r$.

Theorem: Let $S$ be a complete metric space and $(F_n)$ be a sequence of subsets of $S$. If
(i) each $F_n$ is a non-empty closed set,
(ii) $\text{diam}(F_n) \to 0$, and
(iii) $F_m \subseteq F_n$ for each pair of indices $m \geq n$

then there is a unique point in $\bigcap_{n=1}^{\infty} F_n$.

Definition: A set $D$ in a metric space is dense iff $D \cap V \neq \emptyset$ for every non-empty open set $V$.

Baire’s Theorem: If $S$ is a complete metric space and $(U_n)$ is a sequence of open, dense subsets of $S$, then $\bigcap_{n=1}^{\infty} U_n$ is dense.

Definitions: Let $(S,d)$ be a metric space and $A$, $B$, and $C$ denote subsets of $S$.
(a) $A$ is nowhere dense iff every open set $U$ contains an open set $V$ with $V \cap S = \emptyset$.
(b) $B$ is an $F_\sigma$-set iff $B$ is the union of a countable number of closed sets.
(c) $C$ is a $G_\delta$-set iff $C$ is the intersection of a countable collection of open sets.

Baire’s Theorem Restated: If $S$ is a complete metric space and $S = \bigcup_{n=1}^{\infty} F_n$ is the union of a countable collection of closed sets, then at least one of the sets $F_n$ contains an open ball. Put another way, a complete metric space cannot be the union of a sequence of closed, nowhere dense sets.

Corollary: The set of rational numbers in $[a,b]$ is an $F_\sigma$-set but the set of irrational numbers in $[a,b]$ is not an $F_\sigma$-set.

Notation: For a function $f : S \to T$ write $D(f)$ for the set of points in $S$ at which $f$ is discontinuous.
Theorem: For any function \( f : [a, b] \to \mathbb{R} \), \( D(f) \) is an \( F_\sigma \) set.

Corollary: There is no \( f : [0,1] \to \mathbb{R} \) with \( D(f) \) the set of irrationals in \([0,1]\).

Example: Let \( E \) be a countable subset of \([0,1]\) enumerated in some way as \( E = \{ x_n \}_{n \geq 1} \). If \( f(x) = 0 \) for \( x \notin E \) and \( f(x_n) = 1/n \) for each \( n \), then \( D(f) = E \).

Theorem: If \( f_n : [a, b] \to \mathbb{R} \) is a sequence of continuous functions which converges pointwise to \( f \), then \( f \) has a point of continuity in \([a,b]\).

Corollary: If \( f : [a, b] \to \mathbb{R} \) is the pointwise limit on \([a,b]\) of a sequence of continuous functions, then \( \{ x \in [a, b] : f \text{ is continuous at } x \} \) is an uncountable, dense \( G_\delta \) set.

PROBLEMS

Problem 13-1. If \( S \) is a countable and complete metric space then some singleton in \( S \) is an open set.

Problem 13-2. If \( S \) is a complete metric space and \( f_n : S \to \mathbb{R} \) is a pointwise bounded sequence of continuous functions then \( (f_n) \) is uniformly bounded on some open ball in \( S \).

Problem 13-3. In a metric space each compact set is a \( G_\delta \) set.

Problem 13-4 (see Rudin, Remark 4.31, p 97). Let \( E \) be a countable subset of \((0,1)\) enumerated in some way as \( E = \{ x_n \}_{n \geq 1} \). For each \( x \) in \((0,1)\) write \( I(x) = \{ n : x_n < x \} \). Define \( f \) on \((0,1)\) by \( f(x) = 0 \) if \( I(x) \) is empty and \( f(x) = \sum_{n \in I(x)} 2^{-n} \) otherwise. Prove that \( f \) is monotone increasing on \((0,1)\) and \( D(f) = E \).

Problem 13-5. If \( U = (a,b) \) is an interval on the real line, there is a metric \( d(x,y) \) on \( U \) with these properties. (1) \( U \) is complete under \( d \). (2) A sequence \( (x_n) \) converges in \( d \)-metric to a point \( p \) in \( U \) iff it converges to \( p \) in the usual sense. (3) The \( d \)-closed sets are the same as the usual closed sets. (4) The \( d \)-open sets are the same as the usual open sets. (5) Baire's Theorem is true in \( U \) with the usual metric.