APPLICATION OF BAIRE'S THEOREM

Theorem: If \((f_n) \subseteq C(Ta, b] \) and \(f_n \to f \) pointwise on \([a, b] \), then \(f \) is continuous at some point in \([a, b] \).

Corollary: If \((f_n) \subseteq C(Ta, b] \) and \(f_n \to f \) pointwise on \([a, b] \), then \(D(f)^c = \{ x \in [a, b] : f \text{ cont. at } x \} \) is an uncountable, dense \(G_\delta \)-set.

PPCF Theorem: write CBI for "closed, bounded interval."

Step 1: If \((f_n) \subseteq C(Ta, b] \) and \(f_n \to f \) pointwise on \([a, b] \), then \( \forall \varepsilon > 0 \exists \delta \text{ such that } \forall x \in I \subseteq (a, b) \).

\[ \forall x \in I \forall \varepsilon, t > m, \quad |f_{t-1}(x) - f_{t+1}(x)| < \varepsilon \]

Check Step 1: \( F_n = \{ x \in [a, b] : \forall s, t > m, \quad |f_{s-1}(x) - f_{s+1}(x)| < \varepsilon \} \)

\[ = \cap_{s, t} (s, t) \subseteq (a, b, (s, t)) \text{ is the intersection of closed sets and thus closed.} \)

By Baire's Theorem \( \exists m \in \mathbb{N} \) so that \( F_m \) is an open interval \( (p, q) \subseteq I_n \).

Step 2: There are strictly increasing indices \( m(1) < m(2) < m(3) \ldots \) and decreasing CBIs \( I_n = [a_n, b_n] \)
so that (i) \( I_{m(n)} \subseteq (a, b) \) and (ii) \( \forall x \in I_n \forall \varepsilon, t > m, \quad |f_{t-1}(x) - f_{t+1}(x)| < \varepsilon/n \)

Check Step 2: For \( n = 1 \), let \( \varepsilon = 1/1 \) and \( I_0 = [a, b] \).

Using Step 1, \( \exists m(1) \in \mathbb{N} \) so that \( \forall x \in I, \forall t > m(1), \quad |f_{t-1}(x) - f_{t+1}(x)| < \varepsilon/1 \).

Write \( [a_2, b_2] = I \). For the next \( n = 2 \), let \( \varepsilon = 1/2 \) and apply Step 1 on \([a_1, b_1] \).
Again by Step 1, \( f_m(12) > m(11) \implies \forall \epsilon \exists B \subseteq (q, b)
\forall s, t \in m(12), |P_s(x) - P_t(x)| \leq \epsilon/2. \) Write \( I_2 = 1 \) and continue (forever?).

**Step 3:** If \( p \in \bigcap_{n=1}^{\infty} [a_n, b_n] \) then \( P \) is continuous at \( p \). Note the intersection is non-empty because \( I_1 \subseteq (a_n, b_n) < I_2 \) for all \( n \).

Check Step 3. Let \( \epsilon > 0 \), choose \( n \) so that \( 1/n < \epsilon/3 \)

And write \( M = m(n) \). Then

\[ s_j \rightarrow m \text{ and } x \in I_n \implies |P_s(x) - P_m(x)| \leq \epsilon/3 \]

\( P_n \) is uniformly continuous on \( [a, b] \), so

\[ \exists s_1 > 0, |x - z| < s_1 \implies |P_m(x) - P_m(z)| < \epsilon/3 \]

\( P \in I_n \subseteq (a_n, b_n) \implies \exists s_2 > 0, (p - s_2, p + s_2) \cap (a_n, b_n) \subset I_n \)

Let \( s = \min(s_1, s_2) \)

Claim: \( |P - x| < s \implies |P(x) - P_m(x)| < \epsilon/3 \)

Why? \( |P - x| < s \leq s_2 \implies x \in I_n \implies \forall s \leq s_1, |P_s(x) - P_m(x)| \leq \epsilon/3 \).

Letting \( s \rightarrow 0 \),

\[ |P(x) - P_m(x)| = \lim_{s \rightarrow 0} |P(x) - P_s(x)| \leq \epsilon/3 \]

Finally, \( |P - x| < s \implies |P(p) - P_m(p)| \leq |P(p) - P_m(x)| + |P_m(x) - P_m(p)| + |P_m(x) - P_m(x)| \leq 3(\epsilon/3) \), the first two estimates by the claim, the middle estimate by uniform continuity.

**Corollary:** The theorem applies on every interval \([a, b] \implies D(A) = D(A^c) \) is dense. \( D(A) \) is \( F_3 \) for any \( A \).

To see that \( D(A) \) is countable, suppose for a contradiction that it is not. \( D(A) = \bigcap_{n=1}^{\infty} U_n \) for some open, dense \( U_n \) 's.

\( \forall \) also \( D(A) = \bigcap_{n=1}^{\infty} U_n \) then \( I_{n+1} = U_{n+1} \cap U_n \), \( I_n \)

By Baire's Theorem \( \exists V_n \text{ open } V \subseteq U_n \), contradicting density of \( U_m \).