14. THE BROUWER THEOREM FOR THE DISK

Notation: Throughout this handout $D = \{ x \in \mathbb{R}^2 : \| x \| \leq 1 \}$ is the closed unit disk and 
$\partial D = \{ x \in \mathbb{R}^2 : \| x \| = 1 \}$ its boundary. $\partial D$ is oriented counter clockwise. Partial derivatives are denoted by subscripts $A_x$, $P_x$, etc. A function of several variables is called continuously differentiable iff all of its first order partial derivatives exist and are continuous, and is twice continuously differentiable iff all of its second order partial derivatives exist and are continuous.

Lemma (Weierstrass Theorem): If $\phi \in C(D)$ and $\epsilon > 0$ then there is a polynomial $\psi$ in two variables so that $\| \phi - \psi \|_u < \epsilon$.

Lemma: If $f : D \rightarrow \partial D$ is continuous and $\epsilon > 0$ then there is a twice continuously differentiable $g : D \rightarrow D$ with $\| f(x) - g(x) \|_2 < \epsilon$ for all $x \in D$.

Pf: Write $f(x) = (a(x), b(x))$. The coordinate functions $a$ and $b$ are continuous. By the Weierstrass Theorem there are polynomials $\alpha(x)$ and $\beta(x)$ in two variables so that $\|a - \alpha\|_u < \epsilon/2$ and $\|b - \beta\|_u < \epsilon/2$. $h(x) = (\alpha(x), \beta(x))$ is twice continuously differentiable on $D$ and for each $x \in D$,

$\| f(x) - h(x) \|_2 \leq \| f(x) - a(x) \|_2 + \| h(x) - a(x) \|_2 < 1 + \epsilon/2$. For $g$ take $g = (1 + \epsilon/2)^{-1} h$. For each $x \in D$, $h(D) \subset D$ because $\|g(x)\|_2 \leq 1$ and

$\| f(x) - g(x) \|_u \leq \| f(x) - h(x) \|_2 + \| h(x) - (1 + \epsilon/2)^{-1} h(x) \|_2 < \epsilon/2 + (\epsilon/2)(1 + \epsilon/2)^{-1} \| h(x) \|_2 \leq \epsilon$.

Lemma: If there is a continuous $f : D \rightarrow D$ with no fixed point, then there is a twice continuously differentiable $h : D \rightarrow \partial D$ so that $h(x) = x$ for all $x \in \partial D$.

Pf: $D$ is compact and $m(x) = \| f(x) - x \|_2$ is continuous on $D$. $m(x)$ must attain a minimum value and $m = \min \{ \| f(x) - x \|_2 : x \in D \} > 0$ since $f$ has no fixed points. By the preceding lemma there is a twice continuously differentiable $g : D \rightarrow D$ with $\| f - g \|_u < m/2$. $g$ is also fixed point free because for each $x$ in $D$, $m \leq \| f(x) - x \|_2 \leq \| f(x) - g(x) \|_2 + \| g(x) - x \|_2 \leq m/2 + \| g(x) - x \|_2$. Define $b(x) = x + t(x) [x - g(x)]$, where $t(x)$ denotes the unique non-negative scalar for which $\| h(x) \|_2 = 1$. By
the quadratic formula \( t(x) = \frac{-<x, x - g(x)> + \sqrt{<x, x - g(x)>^2 - \|x - g(x)\|^2}}{\|x - g(x)\|^2} \) has non-negative solution

\[
\begin{align*}
-<x, x - g(x)> + \sqrt{<x, x - g(x)>^2 - \|x - g(x)\|^2} & \leq \|x - g(x)\|^2 \\
& \iff -<x, x - g(x)> + \sqrt{<x, x - g(x)>^2 - \|x - g(x)\|^2} \leq 0 \\
& \text{for } x \in \partial D \text{ then } -<x, g(x)> \leq \|x\|^2 \iff -<x, x >, 0 \leq -x, x - g(x), \text{ t}(x) = 0 \text{ and } h(x) = x.
\end{align*}
\]

Computational Review of Line Integrals: Let \( C \subset \mathbb{R}^2 \) be a simple, smooth curve oriented from point \( \tilde{a} \) to point \( \tilde{b} \), meaning \( \exists \sigma : [a, b] \to C \) which is 1-1, onto and continuously differentiable with \( \sigma(a) = \tilde{a} \) and \( \sigma(b) = \tilde{b} \). \( \sigma(t) = (x(t), y(t)) \) is called a parameterization of \( C \) with \( t \) the parameter. Given functions \( P(x,y) \) and \( Q(x,y) \) continuous on \( C \), the line integral \( \int_C Pdx + Qdy \) is calculated by formally substituting

\[
x(t), \ y(t), \ dx(x) = x'(t) \ dt, \ dy(y) = y'(t) \ dt \text{ and then integrating from lower limit } t = a \text{ to upper limit } t = b.
\]

The value of the line integral is independent of the parameterization but not the orientation. Reversing the orientation changes the sign of the line integral.

Illustrative Example: Let \( C \) be the arc of the circle \( x^2 + y^2 = 1 \) which lies in the first quadrant and runs from \( \tilde{a} = (1,0) \) to \( \tilde{b} = (0,1) \). Calculate \( \int_C xy^2 \ dx \) using these two parameterizations.

(a) Let \( x = \cos(t) \) and \( y = \sin(t) \) for \( 0 \leq t \leq \pi/2 \).

\[
\begin{align*}
L &= \int_0^{\pi/2} \cos(t) [\sin(t)]^2 [-\sin(t)] \ dt = \left(\frac{-1}{4}\right) \sin^4(t) \bigg|_{0}^{\pi/2} = -\frac{1}{4}.
\end{align*}
\]

(b) Use \( x \) itself as the parameter and \( y = \sqrt{1 - x^2} \). Integrating from lower limit \( x = 1 \) to upper limit \( x = 0 \) reflects the given orientation of \( C \).

\[
L = \int_1^0 x [\sqrt{1 - x^2}]^2 \ dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right] \bigg|_{1}^{0} = -\frac{1}{4}.
\]

Green's Theorem for the Disk: If \( A(x,y) \) and \( B(x,y) \) are continuously differentiable real valued functions on \( D \) then

\[
\iint_D A_x - B_y \ dx \ dy = \oint_{\partial D} A dy + B dx.
\]

\textbf{Pf:} The integral equality will follow by combining these two separate equalities;

\[
(* \ iint_D A_x \ dx \ dy = \int_{\partial D} A dy \quad \text{and} \quad (** \ iint_D B_y \ dx \ dy = -\int_{\partial D} B dx).
\]

For the first,
\[
\iiint_D A_x \, d x \, d y = \int_{-1}^{+1} \int_{-1}^{+1} \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} A_x (u, y) \, d u \, d y = \int_{-1}^{+1} A (\sqrt{1-y^2}, y) - A (-\sqrt{1-y^2}, y) \, d y
\]
\[
= \int_{-1}^{+1} A (\sqrt{1-y^2}, y) \, d y + \int_{+1}^{-1} A (-\sqrt{1-y^2}, y) \, d y
\]

On the preceding line the first integral is the line integral up the right hand half of \( x^2 + y^2 = 1 \) from \((0, -1)\) to \((0, 1)\), and the second is the line integral down the left hand half of \( x^2 + y^2 = 1 \) from \((0, 1)\) to \((0, -1)\). This proves (*)

For the second,

\[
- \iiint_D B_y \, d x \, d y = - \int_{-1}^{+1} \int_{-1}^{+1} \frac{\sqrt{1-x^2}}{\sqrt{1-y^2}} B_y (x, v) \, d v \, d x = - \int_{-1}^{+1} B (x, \sqrt{1-x^2}) - B (x, -\sqrt{1-x^2}) \, d x
\]
\[
= \int_{+1}^{-1} B (x, \sqrt{1-x^2}) \, d x + \int_{-1}^{+1} B (x, -\sqrt{1-x^2}) \, d x
\]

On the preceding line the first integral is the line integral over the top half of \( x^2 + y^2 = 1 \) from \((1, 0)\) to \((-1, 0)\), and the second is the line integral over the bottom half of \( x^2 + y^2 = 1 \) from \((-1, 0)\) to \((1, 0)\). This proves (**).

**Brouwer Theorem for the Disk:** Every continuous \( f : D \to D \) has a fixed point.

**Proof:** To get a contradiction suppose here is a twice continuously differentiable \( h : D \to \partial D \) which is the identity on \( \partial D \). Write \( h(x, y) = (f(x, y), g(x, y)) \) with \( f \) and \( g \) having continuous first and second order partial derivatives. Define \( J(x, y) = (f \, g_y)_x - (f \, g_x)_y \). Continuity of the second order partials gives \((g_y)_x = (g_x)_y\) and

\[
J(x, y) = (f \, g_y)_x - (f \, g_x)_y = f_x \, g_y + f \, (g_y)_x - f_y \, g_x - f \, (g_x)_y = \det \begin{pmatrix} f_x & g_x \\ f_y & g_y \end{pmatrix},
\]

**Claim (1):** \( J(x, y) = 0 \). Since \((f, g) \in \partial D \, , \, f^2 + g^2 = 1 \, , \, 0 = (f^2 + g^2)_x = 2 \, f \, f_x + 2 \, g \, g_x \) and \( 0 = (f^2 + g^2)_y = 2 \, f \, f_y + 2 \, g \, g_y \). Writing the last two equalities in matrix form, \( \begin{pmatrix} f_x & g_x \\ f_y & g_y \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

The square matrix has a non-trivial kernel and must have determinant \( J(x, y) = 0 \).

**Claim (2):** \( \iint_D J(x, y) \, d x \, d y = \pi \). Parameterize \( \partial D \) by \( x = \cos(t) \) and \( y = \sin(t) \), \( 0 \leq t \leq 2\pi \). By Green’s Theorem

\[
\iint_D J(x, y) \, d x \, d y = \iint_D (f \, g_y)_x - (f \, g_x)_y \, d x \, d y = \iint_D (f \, g_x)_x + (f \, g_y)_y \, d x \, d y
\]
\[
= \int_{-1}^{+1} f \, [g_x x' + g_y y'] \, d t = \int_{-1}^{+1} f(x, y) \, [g(x, y)]' \, d t
\]

On \( \partial D \, (x, y) = h(x, y) = (f(x, y), g(x, y)) \), \( f_x, y) = x = \cos(t) \) and \( g(x, y) = y = \sin(t) \) and
\[
\iint_{D} J(x, y) \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{2\pi} f(x, y) [g(x, y)]' \, dt = \int_{0}^{2\pi} \cos(t) \left[ \sin(t) \right]' \, dt = \int_{0}^{2\pi} \cos^2(t) \, dt = \pi.
\]

The two claims are clearly contradictory.