Problem Set Ten: Comparison Tests

Theorem (Ratio Test): Let \((a_n)\) be a positive sequence for which \(r = \lim \frac{a_{n+1}}{a_n}\) exists.
(a) If \(r < 1\) then \(\sum_{k=1}^{\infty} a_k\) converges.
(b) If \(r > 1\) then \(\sum_{k=1}^{\infty} a_k\) diverges.

Remarks: For \(a_n = n^{-p}\) the Ratio Test gives no information since \(r = 1\). For an exponential sequence \(a_n = c^n\) is \(r = c\). In fact the test is a way to compare a series’ terms with exponentials \(c^n\).

Theorem: Let \((a_n)\) be a positive sequence for which \(r = \lim \frac{a_{n+1}}{a_n}\) exists.
(a) If \(r < 1\) then there are constants \(s\) and \(c\) with \(0 < c, 0 < s < 1\) and \(a_n \leq c s^n\) for all \(n\).
(b) If \(r > 1\) then \((a_n)\) is increasing and can't converge to zero.

In case (a) of the Ratio Test the series converges by comparison with a geometric series and the sequence \((a_n)\) is eventually decreasing.

Theorem (Raabe’s Test): Let \((a_n)\) be a positive sequence for which \(\rho = \lim n \left[\frac{a_n}{a_{n+1}} - 1\right]\) exists.
(a) If \(\rho > 1\) then \(\sum_{k=1}^{\infty} a_k\) converges.
(b) If \(\rho < 1\) then \(\sum_{k=1}^{\infty} a_k\) diverges.

Remark: For a power sequence \(a_n = n^{-p}\) \(\rho = p\); the verification uses the familiar definition \(f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\). Raabe’s Test is a disguised way to compare the series’ terms with powers \(n^{-p}\). We'll do the details later if time permits.

The Ratio and Raabe Tests are special cases of Kummer's Theorem. To derive the Ratio Test from Kummer’s Theorem take \(b_n = 1\); in that case \(\lim c_n = (1/r) - 1\). To derive Raabe’s Test take \(b_n = n\); in that case \(\lim c_n = \rho - 1\).

Theorem (Kummer’s Test): Let \((a_n)\) and \((b_n)\) be positive sequences, write \(c_n = \frac{a_n}{a_{n+1}} b_n - b_{n+1}\) and assume \(\lim c_n\) exists.
(a) If \(\lim c_n > 0\) then \(\sum_{k=1}^{\infty} a_k\) converges.
(b) If \(\sum_{k=1}^{\infty} (1/b_k)\) diverges and \(\lim c_n < 0\) then \(\sum_{k=1}^{\infty} a_k\) diverges.
PROBLEMS

Problem 10-1. Formally rearrange the series on the left to show that \( \sum_{n=1}^{\infty} \left( \sum_{k=n}^{\infty} k^{-p} \right) = \sum_{k=1}^{\infty} k^{1-p} \).

What fact implies that two series have the same sum?

Problem 10-2. Compare each of these to either a geometric or p-series. Which converge? diverge? For each divergent series estimate the size of \( s_n \).

(a) \( \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} \)
(b) \( \sum_{k=1}^{\infty} \frac{1}{k^2} \)
(c) \( \sum_{k=1}^{\infty} \sqrt{k + 1} \)

Problem 10-3 (Limit Comparison Test): Let \( (a_n) \) and \( (b_n) \) be two positive sequences. Prove this Limit Comparison Test using the Comparison Test and basic facts about limits.

(a) If \( L = \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) \) exists and \( \sum_{k=1}^{\infty} b_k \) converges then \( \sum_{k=1}^{\infty} a_k \) converges.
(b) If \( L = \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) \) exists, \( 0 < L \) and \( \sum_{k=1}^{\infty} a_k \) diverges then \( \sum_{k=1}^{\infty} b_k \) diverges.

Problem 10-4. Check for convergence using the Ratio Test.

(a) \( \sum_{k=1}^{\infty} \frac{a^{-k} (k!)^2}{(2k)!} \) for \( a > 0 \)
(b) \( \sum_{k=1}^{\infty} \frac{k^k}{k!} \)

Problem 10-5. Check for convergence using Raabe's Test

(a) \( \sum_{k=1}^{\infty} \frac{1}{(2k)!} \frac{(k+1)!}{(k-2)!} \)
(b) \( \sum_{k=1}^{\infty} \frac{1}{2k (k+1)!} \)
(c) \( \sum_{k=1}^{\infty} \frac{(1)(3)(5) ... (2k-1)}{(2)(4) ... (2k)(2k+2)(4k+1)} \)

Problem 10-6. Suppose \( (a_n) \) is a positive sequence and \( c_n = \frac{a_n}{a_{n+1}} - 1 \leq \frac{1}{2} \) for all \( n \).

(a) If \( \lim_{n \to \infty} c_n \) exists what conclusion can you draw from Kummer’s Test?
(b) Whether or not \( \lim_{n \to \infty} c_n \) exists show there is a constant \( c \) so that \( a_n \leq c \left( \frac{2}{3} \right)^n \) for all \( n \).

Problem 10-7. Suppose \( (a_n) \) is a positive sequence and \( c_n = n \left( \frac{a_n}{a_{n+1}} - 1 \right) \leq 2 \) for all \( n \).

(a) If \( \lim_{n \to \infty} c_n \) exists what conclusion can you draw from Kummer’s Test?
(b) Whether or not \( \lim_{n \to \infty} c_n \) exists show there is a constant \( c \) so that \( a_n \leq \frac{c}{n (n+1)} \) for all \( n \).

Problem 10-8. Show that the Ratio and Raabe's Tests both fail for \( a_n = n \ln(n) \). For that reason \( b_n = n \ln(n) \) is often used for third round of testing with Kummer’s Theorem.
Problem 10-9: The hypergeometric series of Gauss is defined using three parameters $a$, $b$, $c$ and a variable $z$ by

$$F(a, b, c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a+1)...(a+n-1)(b+1)...(b+n-1)}{n!(c+1)...(c+n-1)} z^n.$$

In this problem assume that $z$ is a positive real and that $a$, $b$ and $c$ are positive integers. The series can be written more neatly using the rising factorial notation $(a)_n = (a)(a+1)(a+2)...(a+n-1)$. Note that $(1)_n = n!$ and

$$F(a, b, c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} z^n.$$

(a) Show the series converges when $0 < z < 1$ and diverges for $1 < z$.
(b) In case $z = 1$, show the series converges if $c > a + b$ and diverges if $c < a + b$.
(c) Investigate the case $c = a + b$.

Problem 10-10. For a set of real numbers $A$ the diagram

\[
\begin{array}{ccc}
\text{A is finite} & \rightarrow & \text{A is countable} \\
\downarrow & & \downarrow \\
c(A) = 0 & \rightarrow & \text{meas}(A) = 0
\end{array}
\]

indicates four valid implications. Are there any others? Give proofs or counter-examples.