Problem Set Two: Countable Sets

Definitions: Let \( f : A \to B \).
(a) \( f \) is one-to-one (or 1-1) iff \( \forall x, z \in A, f(x) = f(z) \Rightarrow x = z \) \( a, x \in A \).
   Equivalently, \( f \) is 1-1 iff \( \forall x, z \in A, x \neq z \Rightarrow f(x) \neq f(z) \).
(b) The range of \( f \) is \( \text{Ran}(f) = \{ b \in B : b = f(x) \text{ for some } x \in A \} \).
(c) \( f \) is onto iff \( \text{Ran}(f) = B \), i.e., for each \( b \in B \) there is an \( x \in A \) with \( b = f(x) \).
(d) A function \( f : A \to B \) that is both one-to-one and onto is called a one-to-one correspondence between \( A \) and \( B \).

Definitions: The composition of \( f : A \to B \) and \( g : B \to C \) is the function \( g \circ f : A \to C \) defined by \( (g \circ f)(a) = g(f(a)) \). In case \( f : A \to B \) is both one-to-one and onto, the inverse of \( f \) is the function \( f^{-1} : B \to A \) defined by \( f^{-1}(b) = a \) iff \( f(a) = b \).

Lemma (aka Problem 2-1): If \( f : A \to B \) and \( g : B \to C \) are both onto (respectively, one-to-one) then so is \( g \circ f \). If \( f \) is a one-to-one correspondence then the inverse \( f^{-1} \) is too.

Definition: \( A \) has the same cardinality as \( B \) (in symbols \( A \sim B \)) iff there is a function \( f : A \to B \) that is 1-1 and onto.

Examples: The set of squares \( \{ 1, 4, 9, \ldots, n^2, \ldots \} \) has the same cardinality as the set of natural numbers. The open interval \( (-1,1) \) has the same cardinality as the set of real numbers.

Theorem: "Same cardinality" is an equivalence relation, i.e., has these three properties.
(0, Reflexivity) \( A \sim A \).
(1, Symmetry) If \( A \sim B \) then \( B \sim A \).
(2, Transitivity) If \( A \sim B \) and \( B \sim C \) then \( A \sim C \).

Definitions: (a) \( A \) is finite iff \( A \) is empty or \( A \sim \{1, 2, 3, \ldots, n\} \) for some \( n \).
(b) \( A \) is infinite iff \( A \) is not finite.
(c) \( A \) is countable iff \( A \) is finite or \( A \sim N \).
(d) \( A \) is countably infinite (or denumerable) iff \( A \sim N \).
(e) \( A \) is uncountable iff \( A \) is not countable.

Theorem: Every subset \( A \subset N \) is countable.

Example: The set of prime natural numbers is countable.

Theorem: Let \( C \) be a countably infinite set. For an infinite set \( A \) the following statements are equivalent.
(1) \( A \) is countably infinite.
(2) There is a one-to-one correspondence \( \alpha : C \to A \).
(3) There is a function \( \beta : C \to A \) that is onto.
(4) There is a function \( \gamma : A \to C \) that is one-to-one.

Definitions: The Cartesian product of sets \( A \) and \( B \), denoted by \( A \times B \), is the set of all ordered pairs \( (a, b) \) for \( a \in A \) and \( b \in B \).
Example: $N \times N$ is countable infinite. Why? Use condition (4) of the Theorem with $A = N \times N$ and $C = N$ by checking that the function $\gamma : N \times N \to N$ given by $\gamma(n, m) = 2^n 3^m$ is one-to-one.

Definitions: The union and intersection of sets $A$ and $B$ are, respectively, $A \cup B = \{x : x \in A \text{ or } x \in B\}$ and $A \cap B = \{x : x \in A \text{ and } x \in B\}$. More generally for $\{A_i\}_{i \in I}$ an indexed collection of sets, the union and intersection of the collection are $\bigcup_{i \in I} A_i = \{a : \exists i \in I, x \in A_i\}$ and $\bigcap_{i \in I} A_i = \{a : \forall i \in I, x \in A_i\}$, respectively.

Theorem: (1) Any subset of a countable set is countable.
(2) The Cartesian product of two countable sets is countable.
(3) The union of a countable collection of countable sets is countable.
(4) If $D_1, D_2, \ldots, D_n, \ldots$ is a sequence of countable sets then $\bigcup_{n \geq 1} D_n = \{x : \exists n \in N, x \in D_n\}$ is countable.

Examples: The set of rational numbers is countable (see problems). We'll see later that the set of real numbers is not countable.

Example: If $P$ is the set of all subsets of $N$, then $P$ is uncountable. In fact, if $A_1, A_2, \ldots, A_n, \ldots$ is any list of subsets of $N$, then $W = \{n : n \notin A_n\}$ is not on the list. (It turns out that $P \sim \text{Reals}$.)

PROBLEMS

Problem 2-1. If $f : A \to B$ and $g : B \to C$ are both onto (respectively, both one-to-one) then so is $g \circ f$. If $f$ is a one-to-one correspondence then the inverse $f^{-1}$ is too.

Problem 2-2. Let $c$ be a constant. For $A$ a non-empty set of reals and $B = \{x + c : x \in A\}$ prove that $A \sim B$. Does $A$ have the same cardinality as $D = \{c x : x \in A\}$?

Problem 2-3. Let $A$ and $B$ be non-empty, countable sets of real numbers.
(a) Prove that $S = \{x : x = ab \text{ for some } a \in A \text{ and } b \in B\}$ is countable.
(b) Prove that $S = \{x : x = a + b \text{ for some } a \in A \text{ and } b \in B\}$ is countable.

Problem 2-4. Any open interval has the same cardinality as the set of all real numbers.

Problem 2-5. Read this proof that the set of rational numbers is countable. Supply a reason for each assertion and fill in any missing or unclear steps.
(a) The set of integers is countable.
(b) For a fixed non-zero integer $n$, the set $D_n$ of rational numbers with denominator $n$ is countable.
(c) The set of non-zero rational numbers is the union of the sets $D_n, n \neq 0$.
(d) The set of rational numbers is countable.

The Cartesian product of sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of all $n$-tuples $x = (a_1, a_2, \ldots, a_n)$ with $a_k \in A_k$ for each $k$.

Problem 2-6. The Cartesian product of a finite number of countable sets is countable. Hints; use induction and the fact that $N \times N$ is countable, or modify the proof that $N \times N$ is countable.
Problem 2-7. Here is another proof that \( \mathbb{Q} \), the set of rational numbers, is countable. Define 
\[ \beta : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q} \text{ by } \beta(k, n, m) = (-1)^k (n-1)/m. \] Show that \( \beta \) is onto and explain why that makes \( \mathbb{Q} \) countable.

A polynomial of degree \( n \) is a function of the form 
\[ P(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \ldots + a_n x^n. \] A real number \( z \) is algebraic iff there is a polynomial with integer coefficients \( a_0, a_1, \ldots, a_n \) so that \( P(z) = 0 \).

Problem 2-7. Show that every rational number is algebraic. Show that \( \sqrt{2} \) and \( \sqrt{2} + \sqrt{3} \) are algebraic.

Problem 2-8. In part (b) you'll need the fact that a polynomial of degree \( n \) can have at most \( n \) real roots. 
(a) For fixed \( n \) prove that the set of \( n \)-degree polynomials with integer coefficients is countable.
(b) Prove that the set of algebraic numbers is countable.

Problem 2-9. For \( H \) a non-empty set of real numbers, the rational span of \( H \) is the set of all real numbers of form 
\[ x = \sum_{k=1}^{n} a_k h_k = a_1 h_1 + a_2 h_2 + \ldots + a_n h_n \] for some finite sets \( a_1, a_2, \ldots, a_n \) of rationals and elements \( h_1, h_2, \ldots, h_n \) of \( H \). Prove that if \( H \) is countable then the rational span of \( H \) is countable.

Problem 2-10. Prove that if \( H \) is countable then the rational span of \( H \) is countable.