Problem Set Four: Convergent Sequences

Definitions: A sequence is a function whose domain is the set of natural numbers $\mathbb{N}$. For a real valued sequence $f : \mathbb{N} \to \mathbb{R}$ it is standard to write $a_n$ for the value $f(n)$ and $(a_n)_{n=1}^{\infty}$ or $(a(n))$ for the sequence $f$ itself. In any event $a_n$ or $a(n)$ is the $n$th term of the sequence. Until further notice $(a_n)$, $(b_n)$, etc will denote real valued sequences.

Definitions: $(a_n)$ converges to a real number $a$ iff $\forall \varepsilon > 0 \exists m > 0$ so that $n \geq m \Rightarrow |a_n - a| < \varepsilon$. The number $a$ is the limit of the sequence and $a_n \to a$ or $\lim_{n \to \infty} a_n = a$ is shorthand for "$(a_n)$ converges to $a$". Geometrically $a_n \to a$ means that the terms $a_n$ are eventually in any open interval centered at $a$.

Remarks: (a) $a_n \to a$ iff $(a_n - a) \to 0$. (b) By the Archimedian Principle the number "m" in the limit definition can always be taken to be a positive integer.

Some Basic Examples: (a) A constant sequence $a_n = c$ converges to the constant $c$.
(b) The sequence of alternating signs $a_n = (-1)^n$ doesn’t converge; neither does $b_n = n$.
(c) If $p > 0$ then $n^{-p} \to 0$.
(d) If $0 < c < 1$ then $c^n \to 0$.

Why $c^n \to 0$? Using Bernoulli’s Inequality $(1 + x)^n \geq 1 + nx$ with $1 + x = 1/c$ leads to the inequality $0 < c^n < c \frac{c}{n(1-c)}$. Consequently if $n \geq \frac{c}{\varepsilon(1-c)}$ then $0 < c^n < \varepsilon$.

Theorem: (a) If $a_n \to a$, $b_n \to b$ and $a < b$, then $\exists m \in \mathbb{N}$, $n \geq m \Rightarrow a_n < b_n$.
(b) In particular if $a_n \to a$ and $a < b$, then $\exists m \in \mathbb{N}$, $n \geq m \Rightarrow a_n < b$.
(c) In particular if $b_n \to b$ and $a < b$, then $\exists m \in \mathbb{N}$, $n \geq m \Rightarrow a < b_n$.

Theorem (Sandwich Theorem): Let $(a_n)$, $(b_n)$, and $(c_n)$ be sequence. If $a_n \to p, c_n \to p$ and $\exists m \in \mathbb{N}$, $n \geq m \Rightarrow a_n \leq b_n \leq c_n$, then $b_n \to p$.

Definitions: $(a_n)$ is bounded iff $\exists b > 0 \forall n \in \mathbb{N} \left| a_n \right| \leq b$. $(a_n)$ is a Cauchy sequence iff $\forall \varepsilon > 0 \exists m > 0$ so that $k \& n \geq m \Rightarrow |a_n - a_k| < \varepsilon$.

Theorem: A convergent sequence is Cauchy, and a Cauchy sequence is bounded.

(continued on back)
Definition: For \((a_n)_{n \geq 1}\) a given sequence and \((n(k))_{k \geq 1}\) a strictly increasing sequence of natural numbers, the new sequence with terms \(b_k = a_{n(k)}\) is called a subsequence of \((a_n)\). For example if \(n(k) = 2k\) and \(a_n = 2^{-n}\) then \(b_k = a_{2k} = 4^{-k}\).

Theorem: If \((a_n)\) converges to \(a\) then every subsequence of \((a_n)\) converges to \(a\).

PROBLEMS

Problems 4-1: Which of these sequences have limits? For those with limits verify that the limit definition is satisfied.

(a) \(a_n = \frac{n+10}{2n+\sqrt{n}}\)  
(b) \(b_n = \sum_{k=0}^{n} \frac{1}{(n+k)^2}\)  
(c) \(c_n = \frac{2^n}{n!}\)

Problem 4-2: Let \(r_n\) be the integer remainder when \(n^2\) is divided by 3, e.g., \(r_{13} = 1\) because 13 = 4(3)+1. Without giving formal proofs explain why \((r_n)\) doesn't converge, but give all possible limits of subsequences of \((r_n)\).

In the remaining problems use the limit definition to prove each statement.

Problem 4-3: A sequence cannot have two different limits.

Problem 4-4: If \((a_n)\) is bounded and \(b_n \to 0\) then \(a_n b_n \to 0\).

Problem 4-5: If \(0 < c < 1\) then \(n c^n \to 0\). (Hint; use 1-5.)

Problem 4-6: \(n^{1/n} \to 1\). (Hint; use 1-5.)

Problem 4-7: If \(a_n \to a\), \(b_n \to b\) and \(\exists m \in \mathbb{N}, n \geq m \Rightarrow a_n \leq b_n\), then \(a \leq b\).

Problem 4-8: If \(a_n \to a\) and \(p: \mathbb{N} \to \mathbb{N}\) is one-to-one then \(a_{p(n)} \to a\). In particular if \(m\) is a fixed positive integer and \(a_n \to a\) then \(a_{n+m} \to a\).

Problem 4-9: If \(C\) is any countable collection of sequences, show there is a sequence that has each sequence in \(C\) as a subsequence.