Problem Set Five: Limit Arithmetic

Theorem: Suppose \( a_n \to a \), \( b_n \to b \) and \( c \) is a constant.

(a) \( a_n + b_n \to a + b \)

(b) \( c a_n \to ca \)

(c) \( a_n b_n \to ab \)

(d) If \( b \neq 0 \) then \( b_n \) is eventually non-zero and \( a_n / b_n \to a/b \).

Example: For \( c \) any positive constant, \( c^{1/n} \to 1 \).

PROBLEMS

Problem 5-1: For any reals \( x \) and \( z \), \( |x| - |z| \leq |x - z| \). In consequence if \( a_n \to a \) then \( |a_n| \to |a| \).

Problem 5-2: For any non-negative reals \( x \) and \( z \), \( \sqrt{x} - \sqrt{z} \leq \sqrt{|x - z|} \). In consequence if \( (a_n) \) has non-negative terms and \( a_n \to a \) then \( a \) is non-negative and \( \sqrt{a_n} \to \sqrt{a} \).

Problem 5-3 (Known to the Pythagoreans): There are no pairs of natural numbers with \( 2a^2 = b^2 \) but there are pairs with \( 2a^2 = b^2 \pm 1 \). Recursively define two sequences \( (a_n) \) and \( (b_n) \) like this; \( a_1 = 1 \), \( b_1 = 1 \), \( a_{n+1} = a_n + b_n \) for \( n > 1 \) and \( b_{n+1} = 2a_n + b_n \) for \( n > 1 \). Prove that \( 2a_n^2 - b_n^2 = \pm 1 \) for each \( n \) and that \( b_n / a_n \to \sqrt{2} \).

Problem 5-4: For \( 0 < c < 1 \) and \( p \) a fixed positive integer, \( n^p c^n \to 0 \). (Hint, use 3-5 and the product rule.)

Problem 5-5: If each \( a_n \) is positive and if the sequence of nth powers \( (a_n)^n \) has a positive limit, then \( a_n \to 1 \).

Problem 5-6: (Cesaro’s Theorem): If \( a_n \to a \) and \( (u_n) \) is the sequence of arithmetic averages \( u_n = [a_1 + a_2 + a_3 + \ldots + a_n] / n \), then \( u_n \to a \). Why doesn’t this follow from the sum rule?

Problem 5-7: For \( 0 < a_1 < 1 \) define \( (a_n) \) recursively by \( a_{n+1} = (1 - a_n)^2 \).

(a) Show that if \( a_n \to p \) then \( p = [3 - \sqrt{5}] / 2 \). In the remaining parts \( p \) is the same number.

(b) For any \( n \), \( |a_{n+1} - p| = |a_n - p| [2 - p - a_n] \).

(c) If \( (a_n) \) converges, then \( a_n = p \) for all \( n \).