Problem Set Six: Monotone Sequences

Definition: $(a_n)$ is increasing iff $k < n$ always implies $a_k \leq a_n$, and is strictly increasing iff $k < n$ always implies $a_n > a_k$. $(a_n)$ is bounded above iff there is a constant $b$ so that $a_n \leq b$ for all $n$. The terms decreasing, strictly decreasing, and bounded below are defined similarly. A monotone sequence is one that is increasing or decreasing.

Examples: $a_n = (2n - 1)/(n + 1)$ is strictly increasing and bounded above by 2. $b_n = n 2^{-n}$ is decreasing and bounded below by 0.

Example: For a given $0 \leq a_1 \leq 1$ define $(a_n)$ recursively by $a_{n+1} = \min\{2a_n, 1\}$. Then $(a_n)$ is monotone increasing and bounded above by 1.

**Monotone Convergence Theorem:** If $(a_n)$ is monotone increasing and bounded above, then $(a_n)$ converges to $a = \lim\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$.

Definition: For $(a_n)_{n \in \mathbb{N}}$ a given sequence and $(n(k))_{k \geq 1}$ a strictly increasing sequence of natural numbers, the new sequence with terms $b_k = a_{n(k)}$ is called a subsequence of $(a_n)$. For example if $a_n = 2^{-n}$ and $n(k) = 2k$ and then $b_k = 2^{-2k} = 4^{-k}$.

Lemma: A sequence $(a_n)$ with no largest value has a strictly increasing subsequence.

Proof: The terms of a strictly increasing subsequence can be generated recursively by repeatedly using this Claim: $\forall s \in \mathbb{N} \exists t \in \mathbb{N}$ so that $t > s$ and $a_t < a_s$.

Proof of Claim: For a given index $s$ let $r$ be an index with $1 \leq r \leq s$ and $a_r = \max\{a_1, a_2, \ldots, a_r\}$. Since $a_r$ is not the largest value of the sequence, there is a term $a_t$ so that $a_r < a_t$. Then $a_s \leq a_r < a_t$ and $t > s$ (otherwise $a_t \leq a_r < a_t$).

Building the Subsequence: Let $n(1) = 1$. Using the claim with $s = n(1)$ there is an index $n(2) > n(1)$ with $a_{n(1)} < a_{n(2)}$. Next using the claim with $s = n(2)$ there is an index $n(3) > n(2)$ with $a_{n(2)} < a_{n(3)}$. Continuing in this manner will give a strictly increasing sequence of indices $(n(k))$ for which $(a_{n(k)})$ is strictly increasing.

**Monotone Subsequence Theorem:** Every sequence has a monotone subsequence.

Proof: In the proof $k, s, t$ and $r$ denote positive integers. There are two cases.

**Case One:** Assume the sequence $(a_n)$ has a subsequence with no largest value and let $(b_k) = (a_{n(k)})$ be a subsequence of $(a_n)$ with no largest value. By the preceding lemma $(b_k)$ has a strictly increasing subsequence and that subsequence of $(b_k)$ is also a subsequence of $(a_n)$.
Case Two: Assume every subsequence of \((a_n)\) has a largest value. In this case a decreasing subsequence can be generated recursively by repeatedly using this Claim: \(\forall s \in \mathbb{N} \exists t \in \mathbb{N}\) so that \(t > s\) and \(\forall r \geq t, a_t \geq a_r\).

Proof of Claim: For a given index \(s\) consider the tail subsequence of all terms \(a_n\) with \(n > s\). That subsequence has a term of largest value \(a_t\). \(a_t\) is in the tail subsequence so \(t > s\). All the terms \(a_r, r \geq t\), are also in the tail subsequence and \(a_t\) is the largest value of these tail terms so \(a_t \geq a_r\) for all \(r \geq t\).

Building the Subsequence: Let \(n(1)\) be an index for which \(a_{n(1)}\) is the largest value of the sequence \((a_n)\). By the claim with \(s = n(1)\) there is an index \(n(2) > n(1)\) so that \(a_{n(2)} \geq a_r\) for all \(r \geq n(2)\). Notice \(a_{n(1)} \geq a_{n(2)}\) because \(a_{n(1)}\) is the largest value of the sequence. Next using the claim with \(s = n(2)\) there is an index \(n(3) > n(2)\) with \(a_{n(3)} \geq a_r\) for all \(r \geq n(3)\). Notice \(a_{n(2)} \geq a_{n(3)}\) because \(n(3) > n(2)\) and \(a_{n(2)}\) was chosen to satisfy \(a_{n(2)} \geq a_r\) for all \(r \geq n(2)\). Continuing in this manner will give a strictly increasing sequence of indices \((n(k))\) for which \((a_{n(k)})\) is decreasing.

PROBLEMS

Problem 6-1 (Use Freshman Calculus on this problem): \(e_n = (1+1/n)^n\) is strictly increasing (Problem 1-4).

Here is a short proof that the sequence converges to \(e\). \(\ln (e_n) = n \ln (1+1/n) = \frac{\ln (1+1/n) - \ln (1)}{1/n}\). The last quotient converges to 1. Why is that? Once \(\ln (e_n) \to 1\) is known, \(e_n = \exp (\ln (e_n)) \to \exp (1) = e\).

Problem 6-2: You know that \(e_n = (1+1/n)^n\) is strictly monotone increasing and converges to \(e\). Combine that fact with properties of \(\ln\) to limits to check the following:

\[
\left(1 + \frac{1}{2n}\right)^n \to e \\
\left(\frac{n + 3}{n + 2}\right)^n \to e \\
\left(1 + \frac{2}{n}\right)^n \to e^2 \\
\left(\frac{1}{2} + \frac{2}{n}\right)^n \to 0
\]

Problem 6-3: Show that \(c_n = n^n/(e^n n!)\) is monotone decreasing and bounded below.

Problem 6-4: (a) Prove that if \((a_n)\) has positive terms, is monotone decreasing, and \(a_n = [(n-1)/n]a_{n-1}\) for each \(n\) then \(a_n/a_{n+1} \to 1\).

(b) Show that \(I_n = \int_0^{\pi/2} \sin^n(x) \, dx\) is monotone decreasing. Use part (a) to conclude \(I_{n+1}/I_n \to 1\) and

\[
(\text{Wallis' Formula}) \, w_n = \frac{(2)(4)(6)(8)...(2n)(2n)}{(1)(3)(5)(7)...(2n-1)(2n+1)} \to \frac{\pi}{2}.
\]

(c) Rewrite Wallis' Formula as \(w_n = \frac{4^n (n!)^4}{(2n)!(2n+1)!} \to \frac{\pi}{2}.

Problem 6-5: Define \((z_r)\) recursively by \(z_{n+1} = [(2z_n^2 + 1)/3]\). Prove that if \(z_1 < 1/2\) then \((z_n)\) is monotone increasing and bounded above. Find the limit.
Problem 6-6: Let $s_n = \sum_{k=2}^{n} k^{-2}$. Prove that $(s_n)$ is monotone increasing and bounded above. To find an upper bound you might use the inequality $\frac{1}{k^2} \leq \frac{1}{k-1} - \frac{1}{k}$, $k > 1$.

Problem 6-7: If $(a_n)$ is monotone decreasing and bounded below, then $(a_n)$ converges to $a = \text{glb} \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$.