Problem Set Nine: Basic Series

Definition: For a given sequence \( (a_n) \) let \( s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k \). The sequence \( (s_n) \) is called the series with \( n \)th term \( a_n \) and \( n \)th partial sum \( s_n \). The usual notation is to write \( \sum_{k=1}^{\infty} a_k \) to denote the series with terms \( a_k \). The series converges to \( s \), or the series has sum \( s \), iff \( s_n \to s \). It is traditional to also use \( \sum_{k=1}^{\infty} a_k \) for the sum of a convergent series. A series is divergent iff it does not converge.

Example: For fixed \( a \) and \( c \) the geometric series \( \sum_{k=0}^{\infty} c a^k \) converges iff \( 1 < a \), in which case the sum is \( \frac{c}{1-a} \). The partial sums are \( \sum_{k=0}^{n} c a^k = \frac{c(1-a^{n+1})}{1-a} \) for \( a \neq 1 \).

Theorem: (a) If \( \sum_{k=1}^{\infty} a_k \) converges then \( a_n \to 0 \).
(b) If \( \sum_{k=1}^{\infty} |a_k| \) converges then \( \sum_{k=1}^{\infty} a_k \) converges.

Definition: \( \sum_{k=1}^{\infty} a_k \) is absolutely convergent iff \( \sum_{k=1}^{\infty} |a_k| \) converges. \( \sum_{k=1}^{\infty} a_k \) is conditionally convergent iff it converges but \( \sum_{k=1}^{\infty} |a_k| \) diverges.

Theorem: Let \( \sum_{k=1}^{\infty} a_k \) be a series with non-negative terms.
(a) The sequence of partial sums \( (s_n) \) is monotone increasing.
(b) The series converges iff sequence of partial sums \( (s_n) \) is bounded above, in which case the sum of the series is \( s = \sup \{ s_n : n \geq 1 \} \).

Theorem (Comparison Test): Let \( (a_n) \) and \( (b_n) \) be two non-negative sequences, and suppose that \( \exists c > 0 \exists m \in \mathbb{N} \) so that \( n \geq m \Rightarrow a_n \leq c b_n \). Then
(a) if \( \sum_{k=1}^{\infty} b_k \) converges then \( \sum_{k=1}^{\infty} a_k \) converges, and
(b) if \( \sum_{k=1}^{\infty} a_k \) diverges then \( \sum_{k=1}^{\infty} b_k \) diverges.

Theorem: Let \( f(x) \) be a non-negative and decreasing function defined for \( x \geq m \).
Define \( g_n = \sum_{k=m}^{n} f(k) - \int_{m}^{n} f(x) \, dx \).
(a) The sequence \( (g_n) \) is non-negative, monotone decreasing, and thus convergent.
(b, Integral Test) The series \( \sum_{k=m}^{\infty} f(k) \) converges iff \( \lim_n \int_{m}^{n} f(x) \, dx \) exists, ie, iff \( \int_{m}^{\infty} f(x) \, dx \) is convergent.

Example: For \( 0 < p \) the \( p \)-series \( \sum_{k=1}^{\infty} k^{-p} \) converges iff \( p > 1 \).
Definition: The harmonic series is the divergent series $\sum_{k=1}^{\infty} \frac{1}{k}$. Euler's constant is the limit 
$\gamma = \lim_{n \to \infty} \gamma_n$ of the sequence $\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \int_{1}^{n} x^{-1} \, dx = \sum_{k=1}^{n} \frac{1}{k} - \ln (n)$. 

Example: The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is conditionally convergent and has sum $\ln(2)$. 

PROBLEMS

Problem 9-1. Give the sum of the geometric series $\sum_{k=2}^{\infty} 3 \left(\frac{-1}{4}\right)^{k}$. 

Problem 9-2. Show that $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ converges by comparison to a geometric series. 

Problem 9-3. Do these converge or diverge? $\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1}$ and $\sum_{k=1}^{\infty} \frac{1}{k}$. 

Problem 9-4. For what positive values of $p$ does $\sum_{k=2}^{\infty} \frac{1}{k \ln(k)^p}$ converge? Diverge? For the divergent series give a closed form lower estimate for the $n$th partial sum. (“Closed form” means not involving a summation sign.) 

Problem 9-5. Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be convergent series with sums $A$ and $B$, respectively. For $c$ a constant prove that $\sum_{k=1}^{\infty} (a_k + b_k)$ and $\sum_{k=1}^{\infty} c a_k$ have sums $A + B$ and $cA$, respectively. 

Problem 9-6. A permutation of the natural numbers is a function $p : N \to N$ that is one-to-one and onto. For a given permutation $p$ the series $\sum_{k=1}^{\infty} a_{p(k)}$ is called a rearrangement of $\sum_{k=1}^{\infty} a_k$. Prove that if $\sum_{k=1}^{\infty} a_k$ is a convergent series with non-negative terms, then every rearrangement converges and has the same sum. 

Problem 9-7. Rearrange the terms of the alternating harmonic series so that the $3n^{th}$ partial sum is 
$s_{3n} = \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \left(\frac{1}{9} + \frac{1}{11} - \frac{1}{6}\right) + \ldots \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}\right)$. 
Prove that the series converges to $\frac{3 \ln(2)}{2}$.

Problem 9-8. Rearrange the terms of the alternating harmonic series like this;
\[
\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{5} - \frac{1}{4}\right) + \left(\frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6}\right) + \left(\frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} - \frac{1}{8}\right) + \ldots
\]

Prove that this series diverges.

**Problem 9-9.** Suppose that \((a_n)\) is a positive decreasing sequence and \(\sum_{k=1}^{\infty} a_k\) converges. Show that \(n a_n \to 0\).

**Problem 9-10 (There is no largest convergent series.)** If \(\sum_{k=1}^{\infty} a_k\) is a convergent series with non-negative terms, prove there is an positive, unbounded sequence \((c_k)\) so that \(\sum_{k=1}^{\infty} c_k a_k\) converges.

**Problem 9-11 (There is no smallest divergent series.)** If \(\sum_{k=1}^{\infty} b_k\) is a divergent series with non-negative terms, prove there is a positive sequence \((c_k)\) so that \(c_k \to 0\) and \(\sum_{k=1}^{\infty} c_k b_k\) diverges.

Definition: The **Lebesgue outer measure** (or simply the **measure**) of a set \(A\) of reals is

\[
\text{meas}(A) = \text{glb} \left\{ \sum_{k=1}^{\infty} \text{len}(I_k) \right\},
\]

where the glb is taken over all covers \(C = \{ I_k \}_{k=1}^{\infty}\) of \(A\) by a countable number of open intervals.

Observe that \(\text{meas}(A) \leq \text{con}(A)\).

**Problem 9-12.** Prove these properties of measure. (a) \(A \subset B\) implies \(\text{meas}(A) \leq \text{meas}(B)\).

(b) \(\text{meas} \left( \cup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \text{meas}(A_n)\) for any countable collection of sets.

(c) \(\text{meas}([a, b]) = b - a\).

(d) The measure of any countable set is zero.