INDUCTION

N denotes the set of natural numbers.

$EX \ (Oblong \ Numbers)$ For each natural number $n$,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$ 

Intuitive (ie, Picture) Proof, illustrated with $n = 4$

Count dots

\[
\begin{array}{c}
\vdots \ \vdots \\
:\vdots : \\
\vdots \vdots \\
\vdots \vdots \\
\vdots \\
\end{array}
\]

(4,5) dots = $2 + 4 + 6 + 8 = 2(1 + 2 + 3 + 4)$

\[
\frac{4 + 5}{2} = 1 + 2 + 3 + 4
\]

Induction Proof: The initial statement, $1 = \frac{1(1+1)}{2}$, is true.

For the induction step, assume $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ for $n$.

Certain value $n$. Then

$$1 + 2 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{(n+1)(n+1+1)}{2}.$$ 

$EX \ (Bernoulli's \ Inequality).$ For $n$ a natural number and $x \geq -1$, $(1 + x)^n > 1 + nx$. Further, if equality holds for an $n \geq 2$, then $x = 0$.

Calculus Proof: Consider $y = (1 + x)^n$

$y' = n(1 + x)^{n-1} > 0$ so $y$ is increasing.

$y'' = n(n-1)(1 + x)^{n-2} > 0$ so $y$ is convex (=concave up).

The tangent line at $(0,1)$, which is $y = 1 + nx$, is under the graph.

Induction Proof: The initial step $(1 + x) > 1 + l \cdot x$ is true.

Assume the inequality holds for a certain value $n$.

$$(1 + x)^{n+1} = (1 + x)^n(1 + x) > (1 + nx)(1 + x),$$ since $1 + x > 0$

$$= 1 + (n+1)x + nx^2 > 1 + (n+1)x.$$ 

Further, if the two extreme terms are equal, each inequality is an equality and in particular

$$1 + (n+1)x + nx^2 = 1 + (n+1)x,$$ which implies $nx^2 = 0$. 


Some Set Notation: "x ∈ S" is "x is an element of S."
"A ⊆ B" means each element of A is also an element of B.
∅ denotes the empty set, the set with no elements.

Well-Ordering Principle: If S ⊆ N is any non-empty set, then S has a least element.

I'll use induction to prove the contrapositive statement, which is, "If S has no least element then S = ∅." By induction we'll show "If S has no least element then none of 1, 2, 3, ..., n are in S."

Initial Step: 1 ∉ S, since otherwise, 1 would be the least element of S. Induction Step: Assume S has no least element and, for a certain n, none of 1, 2, ..., n are in S. If n+1 were in S, it would be the least element of S. Thus n+1 ∉ S too.

Functional Notation: f: A → B is shorthand for "f is a function with domain A and values in B."

Problem Set 2 gives the usual definitions of domain, range, one-to-one and onto. In case A and B are sets of numbers, f is strictly increasing iff whenever a ≠ a' are in A with a < a', then f(a) < f(a').

Theorem: If B ⊆ N is infinite then there is a unique f: N → B which is strictly increasing and onto.
Proof of Existence: First note that for each \( m \in \mathbb{N} \), 
\( \{ b \in B : b > m \} \) is non-empty and has a least element, for otherwise \( B \subseteq \{ 1, 2, \ldots, m \} \) would be finite. Define \( f(n) \) one natural number at a time like this: 
\[
\begin{align*}
    f(1) &= \text{least } b \\
    f(2) &= \text{least } \{ b \in B : b > f(1) \} \\
    \cdots & \\
    f(n) &= \text{least } \{ b \in B : b > f(n-1) \} \\
\end{align*}
\]
In a picture: 

\( 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ ) \quad (B \ \text{elements})

\( f(1) \ f(2) \ f(3) \ f(4) \ \text{etc.} \)

Proof of Uniqueness: Let \( f \) and \( g \) be two such functions.
And, to get a contradiction, suppose \( f \neq g \). Then \( \forall n \in \mathbb{N} : f(n) \neq g(n) \) is non-empty; let \( l \) be the least natural number with \( f(l) \neq g(l) \). For convenience assume \( f(l) < g(l) \).

Claim that \( \forall n \in \mathbb{N} : g(n) \neq f(n) \). Consider two cases.
(i) \( n < l \) \( \Rightarrow g(n) = f(n) \) because \( l \) is the least value of \( k \) for which \( g(k) = f(k) \).

But \( n < l \) \( \Rightarrow f(n) < f(l) \) because \( f \) is strictly increasing.
Combining inequalities, \( g(n) < f(l) \).

(ii) \( l \leq n \) \( \Rightarrow g(l) \leq g(n) \) since \( g \) is strictly increasing.

\( f(l) < g(l) \) by assumption, and combining inequalities shows \( f(l) < g(n) \).

Punchline. The claim contradicts the fact that \( g \) is onto, because \( f(l) \in B \) but \( f(l) \) isn't a value of \( g \).
COUNTABLE SETS

Definition: A has the same cardinality as B
(in symbols \( A \sim B \)) if and only if there is a function
\( f : A \to B \) that is 1-1 and onto.

Example: \( S = \{1, 4, 9, \ldots, n^2, \ldots\} \), the set of squares,
has the same cardinality as \( \mathbb{N} \) : \( f : S \to \mathbb{N} \)
The square root function \( f(x) = \sqrt{x} \) is 1-1 and onto.

Example: The open interval \(( -1, 1)\) has the same

cardinality as the set of reals \( \mathbb{R} \). For instance,
\( f(x) = \tan \left( \frac{\pi x}{2} \right) \) maps \(( -1, 1)\) 1-1 and onto \( \mathbb{R} \).

Theorem: "Same cardinality" is an equivalence
relation, i.e., has these three properties

(a, Reflexivity) \( A \sim A \) because \( f(x) = x \) is 1-1 and onto.
(b, Symmetry) \( A \sim B \Rightarrow B \sim A \). Why? \( f_1 \)
\( f : A \to B \) is 1-1 and onto, so is \( f^{-1} : B \to A \) (Problem 2-1).
(c, Transitivity) \( A \sim B \) and \( B \sim C \Rightarrow A \sim C \).

For \( f : A \to B \) and \( g : B \to C \) both 1-1 and onto, the
composition \( g \circ f : A \to C \) is 1-1 and onto (Problem 2-1).

Definitions: (a) \( A \) is finite if and only if \( A = \emptyset \) or
\( A \sim \{1, 2, \ldots, n\} \) for some \( n \in \mathbb{N} \).
(b) \( A \) is infinite if \( A \) is not finite.
(c) \( A \) is countable if \( A \) is finite or \( A \sim \mathbb{N} \).
(d) \( A \) is countably infinite (or denumerable) if \( A \sim \mathbb{N} \).
(e) \( A \) is uncountable if \( A \) is not countable.

Theorem: Every subset \( A \subseteq \mathbb{N} \) is countable.
Proof: If \( A \) is infinite there is a function \( f : \mathbb{N} \to A \)
which is onto and strictly increasing. A strictly
increasing function is 1-1.
Example: The set of prime numbers is countable.

Why? There are infinitely many primes.

Theorem: Let \( C \) be a countably infinite set. For an infinite set \( A \), the following statements are equivalent:

1. \( A \) is countably infinite.
2. There is a one-to-one correspondence \( \alpha: C \to A \).
3. There is an onto function \( \beta: C \to A \).
4. There is a one-to-one function \( \gamma: A \to C \).

Note: "One-to-one correspondence" means a function that is 1-1 and onto.

Proof: We'll check the equivalences first in case \( C = \mathbb{N} \).

1. \( \iff \) (2) is the definition.
2. \( \Rightarrow \) (3) holds because a one-to-one correspondence must be onto, i.e., just take \( \beta = \alpha \).
3. \( \Rightarrow \) (4). Let \( \beta: C = \mathbb{N} \to A \) be onto. For each \( a \in A \), \( \{ n \in \mathbb{N} : \beta(n) = a \} \) is non-empty since \( \beta \) is onto, and thus has a least element \( \gamma(a) \). Then \( \beta(\gamma(a)) = a \) because \( \gamma(a) \in \{ n \in \mathbb{N} : \beta(n) = a \} \).

4. \( \Rightarrow \) (1) Let \( \gamma: A \to \mathbb{N} \) be 1-1 and define \( B = \text{Range}(\gamma) \). \( B \) is an infinite subset of \( \mathbb{N} \), so \( B \supseteq \mathbb{N} \). But also \( A \supseteq B \) (\( \gamma: A \to B \) is 1-1 and onto), so by transitivity \( A = \mathbb{N} \).

Proof for denumerable \( C \). Just as in case \( C = \mathbb{N} \):

1. \( \iff \) (2), (2) \( \Rightarrow \) (3) and (2) \( \Rightarrow \) (4). (continued)
Fix a function \( f : \mathbb{N} \to \mathbb{C} \) that is 1-1 and onto.

(3) \( \Rightarrow \) (1): For if \( f : \mathbb{C} \to \mathbb{A} \) is onto, then \( \beta f : \mathbb{N} \to \mathbb{C} \) is onto and \( \mathbb{A} \sim \mathbb{N} \) by the special case.

(4) \( \Rightarrow \) (1): For if \( f : \mathbb{A} \to \mathbb{C} \) is 1-1, then \( \bar{f} : \mathbb{N} \to \mathbb{N} \) is 1-1, and \( \mathbb{A} \sim \mathbb{N} \) by the special case.

Example: \( \mathbb{N} \times \mathbb{N} = \{ (n, m) : n, m \in \mathbb{N} \} \) is countable.

Proof: Define \( \alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) by \( \alpha (n, m) = 2^n 3^m \).

Quoting the uniqueness of the prime factorization proves that \( \alpha \) is 1-1. More directly, suppose \( 2^n 3^m = 2^a 3^b \), and \( a \neq b \) then \( 2^{n-a} 3^{m-b} \) and an even number would be a power of 3. Similarly if \( n < a \) then \( 2^{a-n} 3^b \) again the contradiction that an even number is a power of 3. Thus the only possibility is \( a = n \), which implies \( 1 = 3^{b-m} \), \( b = m \) and finally \( (n, m) = (a, b) \).

Example: \( \mathbb{Q} \), the set of rationals, is countable.

Proof: Define \( \beta : \mathbb{N} \times \mathbb{N} \to \mathbb{Q} \) by \( \beta (n, m) = \frac{n}{m} \).

If \( n \) is even, and \( \beta (n, m) = \frac{n}{m} \). If \( n \) is odd, \( \beta (n, m) = -\frac{n-1}{2m} \).

For instance, \( \frac{13}{7} = \beta (23, 7) \) and \( -\frac{11}{19} = \beta (23, 19) \).

More generally, if \( p > 0 \) and \( q > 0 \) then \( p/q = \beta (2p, q) \) and \( -p/q = \beta (2p+1, q) \).

This shows \( \beta \) is onto and thus \( \mathbb{Q} \) is countable.
Theorem: (1) Any subset of a countable set is countable.
(2) The Cartesian product $A \times B = \{(a,b) : a \in A \land b \in B\}$ of countable sets is countable.
(3) The union of a countable collection of countable sets is countable.
(4) If $D_1, D_2, \ldots, D_n, \ldots$ is a sequence of countable sets then $\bigcup_{n=1}^{\infty} D_n = \{x : \exists n \in \mathbb{N}, x \in D_n\}$ is countable.

Proof: We'll only look at countable infinite sets. The proofs when some sets are finite are similar.

(1) If $A \subseteq C$ with $C$ countable, the observation that the inclusion function $\iota : A \to C$, $\iota(a) = a$, is 1-1 shows $A$ is countable too.

(2) Let $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$ be 1-1, onto functions.
It's clear that $h : A \times B \to \mathbb{N} \times \mathbb{N}$ given by $h(a,b) = (f(a), g(b))$ is also 1-1 and onto.

(4) & (3) For (4), find $f_n : \mathbb{N} \to D_n$ which is onto.
Define $g : \mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} D_n$ by $(n,m) = f_n(m)$.
$g$ is onto and $\mathbb{N} \times \mathbb{N}$ is countable so $\bigcup_{n=1}^{\infty} D_n$ is too.

(3) is similar.

Example: If $P$ is the set of all subsets of $\mathbb{N}$, then $P$ is uncountable. In fact, if $A_1, A_2, \ldots, A_n, \ldots$ is any list of elements of $P$, then $W = \{n \in \mathbb{N} : n \notin A_n\}$ is not on the list. Why? $W = A_m$ for some $m \in \mathbb{N}$, which implies $m \notin W$, and if $m \notin W = A_m$ then $m \in W$. 

Completeness Axiom

Example: Every real number is an upper bound for the empty set $\emptyset$, but $\emptyset$ has no least upper bound.

Why? There are two possibilities for a real $b$: first, $\forall x \in \emptyset, x \leq b$; second, $\exists x \in \emptyset, x > b$. The second is false, so the first must be true.

Examples: The set $\mathbb{N}$ of natural numbers has no least upper bound. Why? Let's see that if $b$ is an upper bound then $b-1$ is an upper bound. In fact, $\forall n \in \mathbb{N}, n+1 \leq n$ and $n+1 \leq b$, implying $n < b-1$.

Completeness Axiom: Any non-empty set of real numbers that has an upper bound has a least upper bound.

Archimedean Principle: The set of natural numbers $\mathbb{N}$ has no upper bound. If $a$ and $b$ are positive reals, then there is a natural number $n$ so that $a < nb$.

Proof from the Completeness Axiom: If $\mathbb{N}$ were bounded above, it would have a L.U.B. (= least upper bound). An example above shows $\mathbb{N}$ has no L.U.B.

For the second statement, $\frac{a}{b}$ is not an U.B. (= upper bound) for $\mathbb{N}$. Thus $\exists n \in \mathbb{N}, \frac{a}{b} < n$.

Theorem: Any open interval of length greater than one contains an integer.

Proof: For the basic case consider an interval $(c,d)$ with $0 < c < d$ and $1 < d-c$. Let $S = \{ n \in \mathbb{N}: d - n \}$. $S \neq \emptyset$ by the Archimedean Principle, and $S$ has a least element by the Well Ordering Principle. There are four short claims (continued next)
(a) \(d \leq l\) because \(l \in S\)
(b) \(l \neq 1\), because otherwise \(0 \leq c < d \leq l = 1\)
would imply \(d - c \leq 1\).
(c) \(l - 1 \in N\) and \(l - 1 \leq d\). Why? \(l \neq 1\) and \(l - 1 \leq l = \text{least}(S) \Rightarrow l - 1 \neq S \Rightarrow l - 1 \leq d\)
(d) \(c < l - 1\). Why? \(c + 1 < c + (d - c) = d \leq l \Rightarrow c < l - 1\)

Combining (c) and (d), \(l - 1 \in (c, d)\)

The remaining two cases are \(c < 0 < d\) and \(c < 0 < d\), the first of which is obvious and the second of which follows since \((\frac{-d}{2}, c)\) contains an integer.

**Corollary:** Let \(Z > 0\). Every open interval contains a rational number \(c^p Z\).

**Proof:** For \((a, b)\) any open interval, the Archimedean Principle gives a \(q \in N\) with \(Z/(b - a) < q\). Since \(1 < (bq/Z) - (aq/Z)\), the preceding theorem gives an integer \(p\) with \(aq/Z < p < bq/Z\). Thus \(a < (p/q) < b\).

**Definition:** A set \(D \subset I\) is dense in \(I\) if and only if each open interval that intersects \(I\) contains a point in \(D\).

**Corollary:** If \(a < b\) then \(\exists x \in [a, b] \mid x\) is rational\).

**Proof:** Let \((c, d)\) be an open interval intersecting \([a, b]\). For \(Z > 0\) the interval \((\max\{c, a\}, \min\{d, b\})\) contains a rational multiple \(rZ^p Z\) which is in both \((c, d)\) and \([a, b]\). \(rZ\) is rational when \(Z = 1\) and irrational when \(Z = \sqrt{2}\).
**Definition:** The absolute value of $x$ is $|x| = \max \{x, -x\}$

**Theorem:** Absolute value has these properties

(a) $|x-a| < c \iff a-c < x < a+c \iff x \in (a-c,a+c)$
the open interval $(a-c,a+c)$

(b) $|z| = 0 \iff z = 0$

(c) $|zw| = |z||w|

(d, Triangle Inequality) $|z+w| \leq |z|+|w|

**Proof (a):** $|x-a|<c \iff x-a<c$ and $-(x-a)<c$
$\iff x<a+c$ and $a-c<x \iff x \in (a-c,a+c)$

**Proof (b):** $0 = |x| = \max \{x, -x\} \iff x \leq 0$ and $-x \leq 0 \iff x = 0$

**Proof (c):** Consider cases. If, say, $z \geq 0$ and $w \leq 0$,
then $|zw| = z(-w) = -(zw) = |zw|$. The other cases are similar.

**Proof (d):** $z+w \leq |z|+|w|$ and $-(z)+(-w) \leq |z|+|w|$, and $|z+w| = \max \{z+w, -(z+w)\} \leq |z|+|w|$ in either case.
CONVERGENT SEQUENCES

Definitions: \((a_n)\) converges to \(a\) iff

\[
\forall \varepsilon > 0 \exists N > 0 \text{ so that } n \geq N \implies |a_n - a| < \varepsilon.
\]

The number \(a\) is the limit of the sequence and \(a_n \to a\) or \(\lim_{n \to \infty} a_n = a\) is shorthand for "\((a_n)\) converges to \(a\)." Geometrically \(a_n \to a\) means that the terms \(a_n\) are eventually in any open interval containing \(a\). Notes (a) \(a_n \to a\) iff \((a_n - a) \to 0\), (b) By the Archimedian Principle the "\(m\)" in the limit definition can be taken to be \(1\), \(N\).

Example: \(a_n = \frac{2n^2 + n}{n^2 + 12} \to 2\)

To Verify

\[
|a_n - 2| = \left| \frac{2n^2 + n}{n^2 + 12} - 2 \right| = \left| \frac{n - 24}{n^2 + 12} \right|
\]

\[
= \frac{n - 24}{n^2 + 12} \leq \frac{n}{n^2 + 12} \leq \frac{1}{n} \leq \frac{1}{n} = \frac{1}{n}
\]

Thus if both \(n > 24\) and \(1/n < \varepsilon\), then \(|a_n - 2| < \varepsilon\). A suitable \(m\) is \(m = \max\{24, 1/\varepsilon\}\).

Some Basic Examples

(a) The constant sequence \(a_n = c\) converges to \(c\).

Why? \(|a_n - c| = 0 < \varepsilon\) if \(c\).

(b) \(a_n = (-1)^n\) and \(b_n = n\), don't converge.

Why? No open interval of length \(\varepsilon\) can eventually contain both \((-1)^n\) and \((-1)^n+1\). Similarly, if

\[
|a_n - a| \leq \frac{1}{n}, \text{ for all } n \geq m, \text{ then } 3 = |(m+3-a) + |a - m|| \leq |(m+3) - a| + |a - m| < 3
\]
Basic Examples (cont.)

(c) If $\rho > 0$, then $n^{-\rho} \to 0$. Why?

(d) If $a < c < 1$, then $c^n \to 0$. Why? Using Bernoulli's Inequality $(1+x)^n > 1+nx$ with $1+x = \frac{1}{c}$, $(\frac{1}{c})^n > nx$ or $0 < c^n < \frac{1}{n} \int_{\frac{1}{c}}^{1} = \frac{1-c}{nc}$. If $n > m = \varepsilon (1-c)/c$ then $0 < c^n < \varepsilon$.

Theorem (a) If $a_n \to a$, $b_n \to b$ and $a < b$,

then $\exists m \in \mathbb{N}$, $n \geq m \Rightarrow a_n < b_n$.

(b) In particular, if $a_n \to a$ and $a < b$,

then $\exists m \in \mathbb{N}$, $n > m \Rightarrow a_n < b$.

(c) In particular, if $b_n \to b$ and $a < b$,

then $\exists m \in \mathbb{N}$, $n \geq m \Rightarrow a < b_n$.

Proof (a) Put $\varepsilon = (b-a)/2$. In the limit definitions,

$\exists m \in \mathbb{N}$, $n \geq m_1 \Rightarrow |a_n - a| < \frac{b-a}{2} \Rightarrow a_n < \frac{a+b}{2}$

$\exists m_2 \in \mathbb{N}$, $n > m_2 \Rightarrow |b_n - b| < \frac{b-a}{2} \Rightarrow \frac{a+b}{2} < b_n$.

When $n > \max \{m_1, m_2\}$, $a_n < (a+b)/2 < b_n$.

Proofs (b) and (c). For (b) take $b_n = b$ constantly. For (c) use $a_n = a$ constantly.

Theorem (Sandwich Theorem). Let $(a_n)$, $(b_n)$ and $c_n$ be sequences for which $\exists m \in \mathbb{N}$, $n \geq m \Rightarrow a_n < b_n < c_n$. If $a_n \to p$ and $c_n \to p$ then $b_n \to p$ too.

Proof: Let $\varepsilon > 0$. $\exists m_1 \in \mathbb{N}$, $n \geq m_1 \Rightarrow |a_n - p| < \varepsilon \Rightarrow p - \varepsilon < a_n$.

$\exists m_2 \in \mathbb{N}$, $n > m_2 \Rightarrow |c_n - p| < \varepsilon \Rightarrow c_n < p + \varepsilon$. Then $n > \max \{m_1, m_2\}$, $p - \varepsilon < a_n < b_n < c_n < p + \varepsilon \Rightarrow |b_n - p| < \varepsilon$. 
Definitions: \( (a_n) \) is bounded if \( \exists b \geq 0 \: \forall n \in \mathbb{N}, \ |a_n| \leq b \). \( (a_n) \) is Cauchy if \( \forall \varepsilon > 0 \: \exists M > 0 \: \text{so that} \ n, k > M \Rightarrow |a_n - a_k| < \varepsilon \).

Theorem: A convergent sequence is Cauchy, and a Cauchy sequence is bounded.

Proof: Suppose \( a_n \to a \). Given \( \varepsilon > 0 \), \( \exists M \in \mathbb{N} \) so that \( s > m \Rightarrow |a_s - a| < \varepsilon/2 \). Then \( k, n > M \Rightarrow |a_n - a_k| < |a_n - a| + |a_k - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon \).

For the second part, suppose that \( (c_n) \) is Cauchy. Find \( m \in \mathbb{N} \), \( n \leq k > m \Rightarrow |c_k - c_m| < 1 \). Then
\[
|a_n| = |(a_n - a_m) + a_m| \leq |a_n - a_m| + |a_m| < 1 + |a_m|.
\]
Combining these facts, \( |a_n| < (|a_1| + |a_2| + \cdots + |a_m|) + 1 \) for all \( n \).

Definition: For \( (a_n) \) a given sequence and \( (n(k)), k \) a strictly increasing sequence of natural numbers, the new sequence with terms \( b_k = a_{n(k)} \) is called a subsequence of \( (a_n) \). For instance, \( (a_n) = (-1)^n \) and \( n(k) = 2k \), \( b_k = a_{2k} = (-1)^{2k} = 1 \).

Theorem: If \( (a_n) \) converges to \( a \), then every subsequence converges to \( a \).

Proof: Recall that if \( (n(k)), k \) is strictly increasing, then \( n(s) \geq s \) for all \( s \). Let \( \varepsilon > 0 \), \( \exists M \in \mathbb{N} \), \( n, m > M \Rightarrow |a_n - a| < \varepsilon \). Then \( k, m > M \Rightarrow n(k) > n(m) \Rightarrow |a_{n(k)} - a| < \varepsilon \).