MATH 625 Sheet 3

To set up the next problem, here is some "review." Let $t > 0$ and suppose $f : [0,t] \to \mathbb{R}$. The *$p-$variation* of f over $[0,t]$ is

$$\langle f,f\rangle_t^{(p)} = \lim_{||P|| \to 0} \sum_{k=0}^{m-1} |f(t_{k+1}) - f(t_k)|^p,$$

where $P = \{t_0, \ldots, t_m\}$ is a partition of $[0,t] : 0 = t_0 < t_1 < \cdots < t_m = t$ and $||P|| = \max_k (t_{k+1} - t_k)$.

The case $p = 1$ has a special interpretation. We say f has *finite arc length* on $[0,t]$ if $\langle f,f\rangle_t^{(1)} < \infty$. The value of this limit is called the *arc length* of f over $[0,t]$. Do you see why this is reasonable? (Think of polygonal approximations).

The case $p = 2$ is also of interest in stochastic calculus: $\langle f,f\rangle_t^{(2)}$ is called the *quadratic variation* of f over $[0,t]$.

1. In this exercise you will show one dimensional Brownian motion B_t (starting at 0) *does not* have finite arc length over any interval $[0,t]$. Stated another way, the distance traveled during any finite time interval is infinite!

Exactly as in the class example just after Corollary 3.1.8, we can show that the limit defining $\langle B,B\rangle_t^{(2)}$ exists as a limit in $L^2(P)$ with value t. Take this for granted.

a) To get a contradiction, assume the arc length of B_t over $[0,t]$ is finite with positive probability (i.e. there exists a set A such that $P(A) > 0$ and for each $\omega \in A$, $\langle B,B\rangle_t^{(1)}(\omega) < \infty$). Prove that on the set A, the quadratic variation of B_t over $[0,t]$ is zero.

HINT: Since Brownian motion is continuous over $[0,t]$, it is *uniformly* continuous there. Recall f is *uniformly continuous* on $[a,b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|s-t| < \delta, \quad s,t \in [a,b] \quad \Rightarrow \quad |f(s) - f(t)| < \varepsilon.$$

In particular, the *same* δ works for all s and t in $[a,b]$.

b) Using that $\langle B,B\rangle_t^{(2)} = t$ and that $(\langle B,B\rangle_t^{(2)} - t)^2 \geq (\langle B,B\rangle_t^{(2)} - t)^2 I_A$, use part a) to get a contradiction.

2. Page 37/3.1

3. Page 42/3.15

4. Page 38/3.4

HINT on part iii):

1. To show that $\int_0^t rB_r\,dr$ is $\mathcal{F}_t$ measurable, look at Riemann sums.
2. To show that X_t is integrable, use Cauchy-Schwarz: $E[|B_s|] \leq \sqrt{E[B_s^2]}$
3. To help prove the martingale property, use Riemann sums to prove $\int_s^t u(B_u - B_s)\,du$ is independent of $\mathcal{F}_s$.

5. Page 38/3.6