MATH 625 Sheet 6

1. Page 74/5.9 Here $\chi_A = I_A$.

2. Your task is to prove Lemma B2 from class:

Lemma B2 Let M_t be a continuous martingale, started at 0, such that for each real u, the process $\exp(iuM_t + \frac{1}{2}u^2 t)$ is a martingale. Prove M_t is Brownian motion.

HINTS:

1. Use the martingale property to compute $E[e^{iuM_t}|\mathcal{F}_s]$.

2. The *characteristic function* of a random variable X is the function

$$f_X(u) = E[e^{iuX}].$$

   It uniquely determines the distribution of a random variable: if $f_X(u) = f_Y(u)$ for all u, then X and Y have the same distribution. The characteristic function of a normal random variable with mean m and variance σ^2 is known to be

$$f(u) = \exp(imu - \sigma^2 u^2/2).$$

   Notice the characteristic function is very similar to the moment generating function. The characteristic function is superior because it is *always* defined, whereas the moment generating function is not.

   Use the characteristic function to check that the increments of M_t have the appropriate normal distribution.

3. Use the characteristic function to prove independence of the increments of M_t. It is known that random variables $X_1, \ldots, X_m$ are independent iff the joint characteristic function factors as the product of the individual characteristic functions:

$$E[\exp(i(u_1 X_1 + \cdots + u_m X_m))] = \prod_{k=1}^{m} E[\exp(iu_k X_k)]$$

3. Page 18/2.12

4. Page 132/7.10
HINTS: Part a: The function $f(x) = x$ is not bounded, so you cannot apply the Markov Property directly. Use $f_n(x) = xI_{[-n,n]}(x)$ and justify taking limits. Also, your final answer should not involve expectations (since the one to part b does not). For this, take the expected value of the SDE in integral form (justify) to get an integral equation for $E^x(X_t)$. Differentiate both sides to get a differential equation and solve it.

5. Page 132/7.3 You can assume f is continuous; then use Lemma M. HINT: Write

$$E^x[f(X_{t+h})|\mathcal{F}_t] = E[f(xe^{c(t+h)+\alpha B_{t+h}})|\mathcal{F}_t]$$

and then $B_{t+h} = (B_{t+h} - B_t) + B_t$.

6. Page 134/7.16