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On The Simulation of Heterogeneous Petroleum Reservoirs

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1. Introduction

The study of petroleum reservoirs is characterized by strongly nonlinear equations, complex physical and chemical processes, strong spatial variation or discontinuities in key reservoir parameters, uncertain or statistical geological data and unstable fluid regimes. Numerical simulation is one of the accepted methods for the study of petroleum reservoirs and improvements in numerical methods is one route which may allow progress in such studies. No single method or set of numerical ideas is sufficient at the present time. In fact computational simulation is used for many distinct length scales, and to suppress or represent accurately a wide range of details in the reservoir and fluid description. The appropriate numerical method then depends on the level of description required and the purpose of the computation. Similarly, we believe that a variety of improved methods might each be useful, possibly in distinct contexts or facets of the reservoir simulation problem. In the same vein,

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we believe that for any proposed idea or method, reservoir parameters and computational problems can be found for which it is ill-suited.

The authors and coworkers have proposed [2, 10, 11, 13, 12, 8] the front tracking method as useful in applications to petroleum reservoir simulation. A variety of tests of a numerical analysis nature were performed for the method, verifying convergence under mesh refinement and absence of mesh orientation effects [13]. The ability to handle complex interface bifurcation [8], fingering instabilities [9, 13] and polymer injection [3, 4] (as an example of tertiary oil recovery) indicates a level of robustness in this method. The main purpose of this paper is to report on two features which will allow further series of tests by enabling a more realistic description of reservoir heterogeneities.

The reservoir equations and the numerical method. The equations governing the flow of two incompressible phases (oil and water) in a porous medium can be approximated by

$$(\phi\, s)_t + \nabla \cdot \mathbf{v}\, f(s) = 0\,, \tag{1.1}$$

$$\mathbf{v} = -\,\mathbf{K}\lambda \cdot \nabla P\,, \tag{1.2}$$

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{K}\lambda \cdot \nabla P = 0\,, \tag{1.3}$$

where $s = s(\mathbf{x},t)$ is the fractional volume of the water phase, $\phi(\mathbf{x})$ is the porosity of the medium, $\mathbf{v}(\mathbf{x},t)$ is the total fluid (oil plus water) velocity, $P(\mathbf{x},t)$ is the pressure, $\mathbf{K}(\mathbf{x})$ is the porous medium (rock) permeability tensor, $\lambda = \lambda(s)$ is the total fluid relative transmissibility function, and $f(s)$ is the so-called fractional flow function relating the water phase velocity to the total fluid velocity. Equations (1.1) and (1.3) represent conservation of the fluids, (1.2) is Darcy's law. We shall consider flow in two spatial dimensions. In order to concentrate on specific physical questions, we have omitted the effects of gravity, capillary pressure (surface tension), variable medium depth, and flow sources from (1.1) through (1.3). See [24, 25] for further details.

There are two flow regimes associated with the form of $f(s)$. The case of a fractional flow function linear in s describes miscible flow; a non-linear function describes immiscible flow. For miscible flow the fluid discontinuities (shocks) are actually contact discontinuities and the shock propagates at the fluid particle velocity. This is not the case for immiscible flow and in that case the fluid particles pass through the shock front. As a consequence, though the hyperbolic equation (1.1) has a simpler wave structure for miscible flow, it is inherently a much more unstable flow regime then immiscible flow. In the linearized (small perturbation) regime, the stability of a jump discontinuity for (1.1) - (1.3) is shown to be determined by the frontal mobility ratio

$$m \equiv \frac{\lambda(s_b)}{\lambda(s_a)} \tag{1.4}$$

where s_a (s_b) is the state on the ahead (behind) side of the traveling shock. The case $m = 1$ is the limit of linear stability, with $m > 1$ corresponding to the unstable regime. For miscible flow m has the potential of becoming infinite, for immiscible flow it is bounded by a

constant value.

The analyses in this paper are sparked
reservoir flow problems. The front track
adaptive methods for the enhanced resoluti
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constant value.

The analyses in this paper are sparked by the application of the front tracking method to reservoir flow problems. The front tracking method is a hybrid which combines special adaptive methods for the enhanced resolution of discontinuities with conventional differencing schemes for the solution in the region between discontinuities. The method as currently implemented is sequential; the pressure (elliptic) equation (1.3) and the conservation (hyperbolic) equation (1.1) are solved separately every time step, each solution involving data from the previous solution of the other. We note that the difference in characteristic time scales between (1.1) and (1.3) provides sufficient justification for this splitting of the system (1.1), (1.3). Employing a sequential method has the distinct advantage of allowing different solution techniques to be used for each equation. The pressure equation is solved by the method of finite elements, which is well suited to the mathematical character of elliptic equations. Similarly a combination of finite differences and analytical Riemann problem solutions is used for (1.1), which is appropriate in view of its hyperbolic nature.

The presence of phase or other types of discontinuities in the physical problem gives rise to discontinuous coefficients in the elliptic pressure equation. A special adaptive grid, which is modified at each time step, is used to resolve these features and compute the flow field accurately [23] . Each such adaptive grid, which we shall refer to as the 'elliptic grid', has the index structure of a regular rectangular $N_e \times M_e$ grid and hence the numerical solution can be accelerated by fast solution techniques.

In the hyperbolic equation, the discontinuities have the mathematical structure of shock waves, and are propagated by jump relations which relate the shock speed to the magnitude of certain discontinuities across the shock. This part of the computation can be viewed as a hybrid of finite differences and of the method of characteristics or of moving point methods [5]. For a discussion of front tracking in greater depth, see [2,13,14] . The solution of (1.1) in the regions between fronts uses a finite difference scheme with respect to a regular rectangular $N_h \times M_h$ grid which covers the entire computational region, and which shall be referred to as the hyperbolic grid.

In the next section, we describe the use of front tracking to represent geological discontinuities such as layers or faults. This material is a preliminary report on a portion of the Ph.D. thesis of one of us (M.M.). Next we report on the effect of porosity variation on reservoir fingering. The main conclusion is that porosity is less significant than permeability as a cause of interface instabilities. A final section contains some comments on M. Shearer's no-go theorem.

2. Geological Layers

2.1. Introduction

In this section we study the effects of a discontinuous permeability tensor, **K**, brought about by the presence of distinct geological layers. We concentrate on the problem of a single discontinuity (a zero width transition zone) separating two homogeneous rock layers of different, constant, permeabilities. We assume the porosity of both layers is the same, and constant, and scale it out by redefining the time variable, t. We are interested in obtaining an analytical understanding of the propagation of an oil-water phase bank (the front) through this geological discontinuity, and in the subsequent development of a numerical algorithm to embody the analytical results and allow calculations through media of greater variation in layering. The point of interaction between a moving front and the stationary rock discontinuity will be referred to as the "node". In the version reported on here, the rock discontinuity shall be assumed to be either horizontal or vertical.

Such an interaction falls into a much broader class of interacting discontinuities of hyperbolic systems. In such interactions one is interested in the general problems of bifurcation, deflection and evolution of the intersection point. Glimm and Sharp [6] studied this phase bank - layer discontinuity interaction as an example of elementary waves and classified the possible exact solutions for the deflection of a front by such a rock discontinuity. Assuming finite, leading order data, they found two solutions. The first consisted of a one parameter family of solutions in which the flow is parallel to the front and the node remains stationary in space. The second is a solution in which flow is normal to the front, and hence the node propagates in space. In this case, the angle of incidence for the front on the rock discontinuity is restricted to a fixed angle given in terms of the ratio of permeabilities for the two geological layers. Both of these solutions are too restrictive to give an indication of how an interaction between the front and the rock discontinuity develops, though they give possible insight into steady state solutions. A third solution was also obtained which allowed the shock to cross the layer tangentially.

This problem has been analyzed with greater generality [21]. The theoretical solution of this problem for general angles and boundary conditions is complicated by the fact that the elliptic equation has a singularity at the node (the velocity is (usually) either zero or infinity). Nevertheless an approximate deflection law can be obtained by introducing an averaging length scale on the scale of the ignored physical phenomena, e.g. capillarity. The resulting equations will be discussed on another occasion [21]. In the next section we describe a simple algorithm that will give an approximate method for deflection, evolution, and lateral bifurcation of a front as it passes a layer. We make two simplifying assumptions.

1) The fractional flow function is the same in both layers (i.e. the same in both rock types).

2) The flow is miscible.

The node propagation routine

In the front tracking method the pr
sequential scheme) is generally accomplis
perpendicular directions in a local coordin
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point of interaction of fronts (a node) the
problem is inherently two dimensional a
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An algorithm for the propagation
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rock discontinuity (i.e. from the upstream
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not represent a moving fluid particle but
tangential slip.

2.2. Cross flow

As an application of the algorithm o
occurs when the flow is mainly parallel to
Fig. 2.1a . Assuming that the left layer is
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Cross flow between the layers can be take
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unstable flow regime ($m > 1$) these press
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for $y > c$ the cross flow would be into t
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represented by Fig. 2.1c, it can be shown t

$$\frac{c - a}{b - a} = \ldots$$

where L is the length of the computational

In the front tracking method the propagation of a front (in the hyperbolic step of the sequential scheme) is generally accomplished by splitting the hyperbolic operator along perpendicular directions in a local coordinate frame. The front is advanced by solving a Riemann problem in the direction normal to the front and the state variable (in this case the saturation) is first updated according to the normal movement of the front. This step is followed by solving the hyperbolic equation, in a direction locally tangential to the front. This is done by a (one dimensional) finite difference scheme, thus updating the state variable at the points which define the front, and accounting for flow tangential to the front. Near a point of interaction of fronts (a node) these steps cannot be taken in the usual way since the problem is inherently two dimensional and splitting into local normal and tangential directions is no longer well defined.

An algorithm for the propagation of the front in the vicinity of a layer has been developed and is given below. Consider a point on the front that is to be propagated past the rock discontinuity (i.e. from the upstream side (layer) to the downstream side (layer) of the discontinuity). The algorithm divides the propagation into two parts. First the point is propagated *using the component of the velocity normal to the front* until it reaches the discontinuity. Then an angle of deflection, velocity and the new normal direction for the propagation into the downstream layer is computed *using only the information in the upstream layer.* The point is then propagated for the remaining time of the timestep using the new propagation direction and velocity. The deflection of the front direction at the discontinuity is analogous to Fermat's principle. We re-iterate the obvious, namely that the point in question does not represent a moving fluid particle but a point on a shock surface along which there is tangential slip.

2.2. Cross flow

As an application of the algorithm discussed above, we study a case of interest which occurs when the flow is mainly parallel to the layer. Consider a miscible flow initialized as in Fig. 2.1a . Assuming that the left layer is of higher permeability than the right, and assuming no cross flow between the layers, at a later time the solution is shown in Fig. 2.1b . Cross flow between the layers can be taken into account qualitatively by noting that the pressure distribution in the no-cross flow solution is piecewise linear in each layer. For the unstable flow regime ($m > 1$) these pressure distributions are shown in Fig. 2.1c . Thus for $y < c$ the pressure distributions would favor cross flow into the high permeability layer and for $y > c$ the cross flow would be into the low permeability layer. (See [29].) Under the approximation that in the cross flow case the pressure distributions can be accurately represented by Fig. 2.1c, it can be shown that

$$\frac{c - a}{b - a} = \frac{1}{1 + m\left(\frac{L}{a} - 1\right)} , \qquad (2.1)$$

where L is the length of the computational region in the y direction.

Fig. 2.2 shows the numerical solution of this problem using the algorithm developed above. For our choice of parameters the ratio (2.1) is about 0.1 . The direction of flow agrees with above approximation; over the major portion of vertical part of the front the cross flow is toward the slow region. Therefore one expects the cross-over point C to be much closer to the slower part of the front as shown in Fig. 2.2 . One might expect that the portion of the front between A and C should move into the fast region, as shown in Fig. 2.1d, but this was not observed numerically. Even when the front was initialized as in Fig. 2.1d, the front did not persist in this configuration.

The finger formation is an indication of a singularity at the node. Initially the singularity becomes stronger as the finger becomes sharper at B, thereby accelerating the runaway behavior. At A, the reverse happens; as the angle of the front with the discontinuity moves away from the normal, the velocity singularity becomes weaker.

3. Porosity Variation

3.1. Introduction

One of the main objectives in oil recovery is to suppress the fingering and channeling instabilities which are initiated by small and large scale disturbances through the nonuniformities in the medium. The nonuniformities that we have in mind are the variations in the permeability and the porosity of the medium. In [3] the fingering problem associated with heterogeneity in the permeability field was studied. See also [20] for a discussion including the effects of capillary pressure. A partial remedy to this effect through the use of polymer flooding was analyzed in [4]. In this section we address the fingering problem associated with variable porosity.

To gain an insight into the effect of porosity variation, consider (1.1) - (1.3). The effect of porosity can be qualitatively understood as follows. If the porosity were constant, it can be removed from (1.1) (leaving (1.2) and (1.3) unchanged), by redefining the time variable, $t \to \bar{t} \equiv t / \phi$. As a consequence, the speed of any wave that would appear in the solution to the hyperbolic equation (1.1) would be modified by a constant factor inversely proportional to the porosity. In the case of variable porosity the above argument can be applied locally, to leading order approximation, thus implying a spatial variation of wave speeds. As in the case of nonuniform permeability, this variation can act to produce fingering.

If we introduce a new velocity field, $\tilde{v} = \frac{v}{\phi}(x)$, (1.1) - (1.3) can be rewritten as

$$s_t + \nabla \cdot (\tilde{v} f(s)) = f(s) \nabla \cdot \tilde{v} , \tag{3.1}$$

$$\tilde{v} = - \frac{K}{\phi} \lambda \cdot \nabla P , \tag{3.2}$$

$$\nabla \cdot K\lambda \cdot \nabla P = 0 . \tag{3.3}$$

From (3.2) we see that the effective rock permeability for the velocity field $\tilde{v}$ is K / ϕ. It is

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Expressing (3.4) in terms of the conserved

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well known that regions of local maxima in the permeability field serve as nuclei for finger growth in porous media flow (see, for example, [3]). Thus (3.2) implies that regions of low porosity will have an effect similar to that of high permeability. However, since the porosity does not enter into the elliptic equation (3.3), its effect will be milder than variations in the permeability field itself. In addition we note that regions of high reservoir porosity commonly have high permeability. Thus porosity variations will normally provide a partial offset to the fingering tendency of permeability variations. .

To explore further the effects of permeability, we consider the equations (1.1)-(1.3) in one spatial dimension. This is particularly appropriate when one thinks of the movement of discontinuities of the hyperbolic solution as a two step procedure (as in the algorithm used in the front tracking method), namely the propagation of the discontinuity in a locally normal direction followed by a step in which the tangential slip of the fluid along the interface is accounted for. In one space dimension, the seepage velocity v is constant as seen from elliptic equation (1.3) and setting it to unity, without any loss of generality, reduces (1.1) to

$$(\phi s)_t + f(s)_x = 0 \ . \tag{3.4}$$

Expressing (3.4) in terms of the conserved quantity ϕs,

$$(\phi s)_t + \frac{f_s}{\phi}(\phi s)_x = -\frac{f_s}{\phi}\, s\, \phi_x \ , \tag{3.5}$$

reveals that, along the characteristic lines

$$\frac{dx}{dt} = \frac{f_s}{\phi(x)} \ , \tag{3.6}$$

ϕs is not constant but changes by

$$\frac{d(\phi s)}{dt} = -\frac{f_s}{\phi}\, s\, \phi_x \ . \tag{3.7}$$

Indeed, the effect of the source term is to force the saturation, s, to be constant along characteristics, as can be seen by expressing (3.4) in the non-conservative form

$$s_t + \frac{f_s}{\phi(x)} s_x = 0 \ . \tag{3.8}$$

Clearly, the variable porosity affects the curvature of the characteristics in space time (equation (3.6)) and the "time to shock formation" when starting with smooth data. It is also seen from (3.6) that the characteristics will not be smooth at a point of discontinuity in ϕ.

In [22], source terms such as found in (3.5) are seen to lead to additional standing waves in the solution of the one-dimensional Riemann problem associated with a hyperbolic equation. Such waves do not arise in the present case due to the special form of the source terms and, in particular, the constancy of s along characteristics.

3.2. Code modification for inclusion of permeability effects

As the variable porosity enters into the system only through the hyperbolic equation (3.1), the adaptation of the front tracking method to the case of variable porosity requires incorporating its effect in the solution of the hyperbolic equation only. Before we discuss the incorporation of the porosity, it will be helpful to describe briefly the basic ideas behind solving the hyperbolic equation in our front tracking method. At any fixed time, the (bounded) spatial domain of the computation consists of a number of regions in which the solution is smooth. These regions are separated by discontinuities across which the saturation is discontinuous. The numerical algorithm to advance the solution of the hyperbolic equation (3.1) from time t to t + dt is done by a spatial splitting of the hyperbolic operator, solving separately for the propagation of the discontinuities (the "front") which includes the solution for the saturation immediately on each side of the discontinuities, and for the solution in the smooth "interior" regions.

The position and shape of each discontinuity is resolved by a finite number of points. Each point of a discontinuity is advanced in a direction (locally) normal to the discontinuity by solving the Riemann problem associated with the normal component of (3.1). The Riemann problem solution gives both the saturation information immediately ahead and behind the propagated point, and its speed of propagation. The solution for these saturations is dictated solely by the flux function, f(s). Since this flux function does not depend on the porosity, the states across the discontinuity are unaffected by the variation in porosity. However, as shown above, the speed of propagation is proportional to the inverse of the local porosity. Inclusion of porosity effects in this step thus consists of incorporating the porosity in the speed of each front point.

The tangential flow of fluid along each discontinuity is incorporated through the solution of the tangential component of the equation (3.1). This is accomplished by employing standard one dimensional finite difference methods (i.e. upwind) using the stencil determined by the representative points. This finite difference solution is obtained separately for each side of the discontinuity, and these solutions are easily modified to take variable porosity into account.

The solution of the two dimensional hyperbolic equation in the smooth regions uses spatial x-y operator splitting and standard one-dimensional finite difference schemes which, as just mentioned, are easily modified to take into account the variation of porosity.

3.3. Test problems

We compare the effects of a variation in the porosity and in the rock permeability fields on fingering in an areal, quarter five-spot flood. Let $\chi(x, y)$ be a gaussian random variable of mean one, and standard variation σ. Let ϕ and K be constant values for rock porosity and permeability respectively. Then $\phi\,\chi(x, y)$ is a gaussian random field for the porosity, with mean ϕ and variation $\sigma\,\phi$. A similar statement holds for $K\,\chi(x, y)$. Using a particular choice for the random variable that allows a specification of a given length scale for the

variation (see [20]), we consider two calc
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4. Shearer's Theorem

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variation (see [20]), we consider two calculations for the unstable flow regime of immiscible displacement; the first with a variation in the porosity field given by $\phi\ \chi(x, y)$ with K fixed, the second with ϕ constant and the variation in the permeability field given by K $\chi(x, y)$. Fig. 3.1 shows the growth of fingers due to the variation in porosity. The local maxima and minima in the porosity field are shown respectively by + and − signs. The fingering can be seen to be initiated in regions of local minima of the porosity. Fig. 3.2 shows the effect of variable permeability with fixed porosity. The permeability variation causes much stronger fingering than the porosity variation.

4. Shearer's Theorem

A mathematical analysis [16,17,26,27] of the equations for three phase (oil, gas, water) flow has revealed a very complex pattern of nonlinear waves and wave interactions. The no-go theorem of M. Shearer states that under very general hypotheses, the "worst case" complications occur generically. This requires a careful examination of both the hypotheses and the conclusions. In this section we give a preliminary analysis of the conclusions.

An elliptic region must occur generically in the three phase flow equations, according to Shearer's theorem. Since the Cauchy problem is ill-posed for elliptic equations and since the Cauchy problem must be solved, an elliptic region could be regarded as a deficiency in the equations. Here we adopt an opposite point of view, and argue that one can learn to live with the elliptic regime [7].

A mathematical analysis of the Riemann problem for related equations reveals no pathology or nonphysical behavior in the solution [15]. A numerical solution of three phase flow equations reaches the same conclusion [1]. The elliptic region is a bounded, interior subset of the state space. The elliptic instability appears to manifest itself by causing the solution to exit from this region, whereupon it enters the (stable) hyperbolic region. Thus the equations could be viewed as predominantly hyperbolic with a non-infinitesimal, nonlinear stabilization of their infinitesimally unstable elliptic region.

In two and three space dimensions, we expect this hyperbolic stabilization to behave in the manner of a phase transition. The fluid will prefer to flow in hyperbolic portions of state space. If an elliptic concentration of phases were somehow initialized, we expect the solution would segregate itself into spatially coherent blobs of mixtures, with each blob located within the hyperbolic portion of phase space. For a first order phase transition described by a single order parameter, the coherent blobs (pure phases) in state space at the edge of the mixed phase region are joined pairwise by tie lines. The tie lines then uniquely sweep out the mixed phase region, and any point in the mixed phase region decomposes into the pure phases at the end of the tie line it lies upon.

In the present case, one would look for tie lines by finding a unique solution of the Riemann problem. However the reasoning is circular, as the Riemann problem is probably not unique in exactly this region. In this case the nonuniqueness is resolved by appeal to more fundamental equations, including capillary pressure diffusion equations, and finally tie lines (or some more complex solution behavior) will be determined as a consequence. It would be most useful to derive an associated order parameter.

Since the elliptic region has little effect on the solution, it is tempting to "eliminate" it by deforming the equations within a fixed topological equivalence class. What are the resulting hyperbolized model equations? Evidently if there are tie lines, they should each be shrunk to a point, leaving a line of umbilic points where the two hyperbolic wave speeds coincide. In this case the hyperbolic behavior in a neighborhood of the elliptic region will coincide with that studied for a polymer flood oil reservoir [19,28,18]. This latter observation has been made previously by B. Keyfitz. However for parameters typical of real reservoirs, the elliptic region is very small. Thus the above line of degenerate hyperbolic points can, in a further approximation, be shrink to a point. Doing this leads an isolated point of degeneracy, as in the models studied by Marchesin and coauthors.

5. Conclusions

The front tracking code developed by the authors and coworkers has been subjected to a series of tests in the petroleum reservoir application. These tests are continuing and are becoming increasingly representative of realistic engineering practice. Fundamental progress in numerical algorithms (e.g. the front untangling and bifurcation algorithms [8]) and in mathematical theory (e.g. the solution of Riemann problems in one and two space dimensions) was necessary for the success of these tests.

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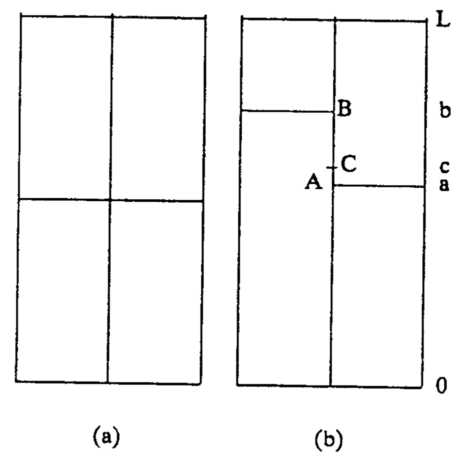

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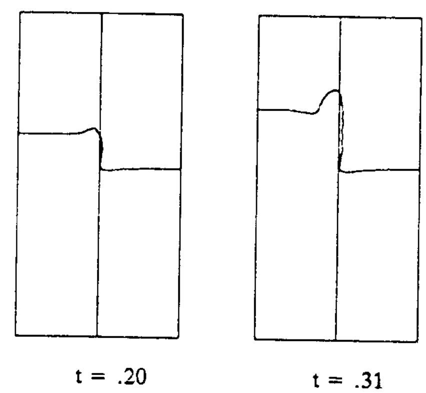

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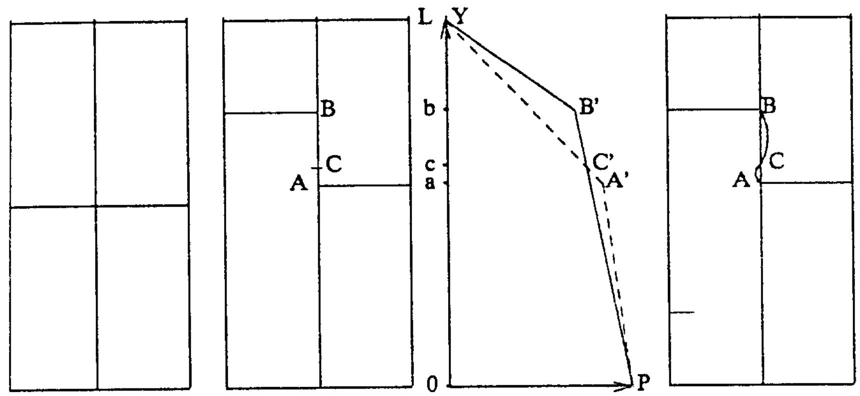

(a) (b) (c) (d)

Fig. 2.1 (a) The initial setup for a miscible flow run through two rock layers separated by a sharp boundary. The initial oil-water bank is horizontal, the higher permeability layer is on the left. (b) The (one dimensional) solution assuming no flow between the layers. (c) The pressure solutions for (b). The solid line is for the left layer, the dashed line for the right. (d) The direction of flow expected from (b) and (c) if cross flow is included.

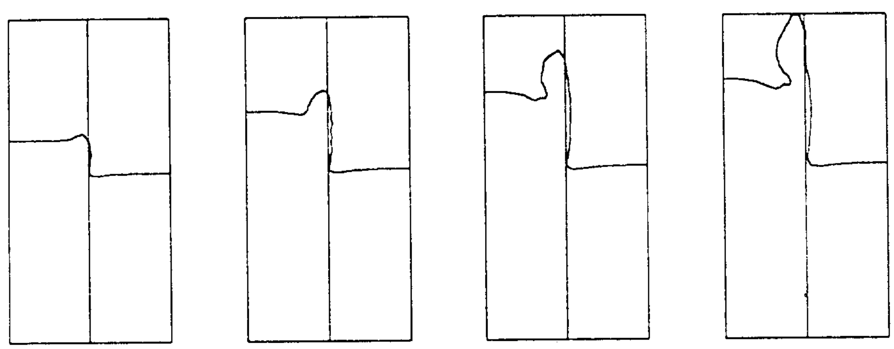

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Fig. 2.2 The solution for the problem discussed in Fig. 2.1 computed using the Front Tracking Method.The initialization is same as in Fig 2.1(a).

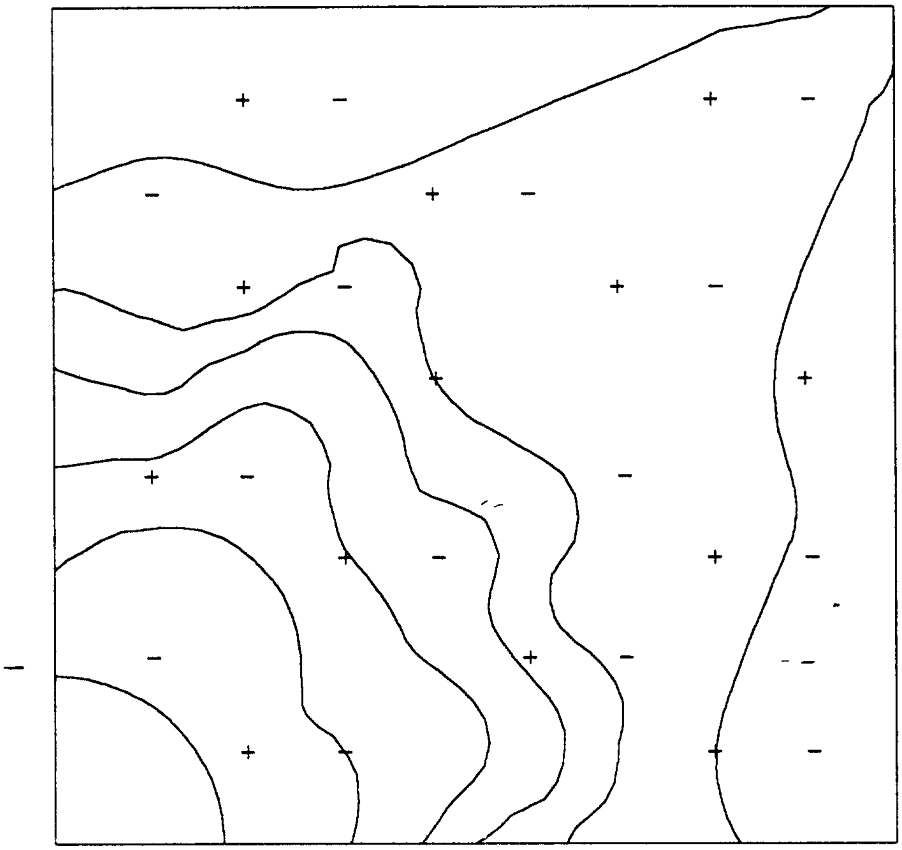
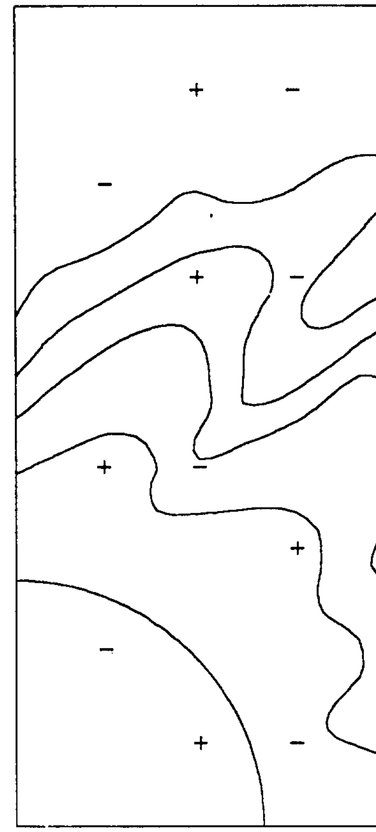


Fig. 3.1 Growth of fingers for immiscible displacement in a heterogeneous reservoir: the temporal evolution of the oil-water discontinuity. The + (−) signs correspond to local maxima (minima) of the porosity field. The viscosity ratio of the water to oil is 1 : 10 .



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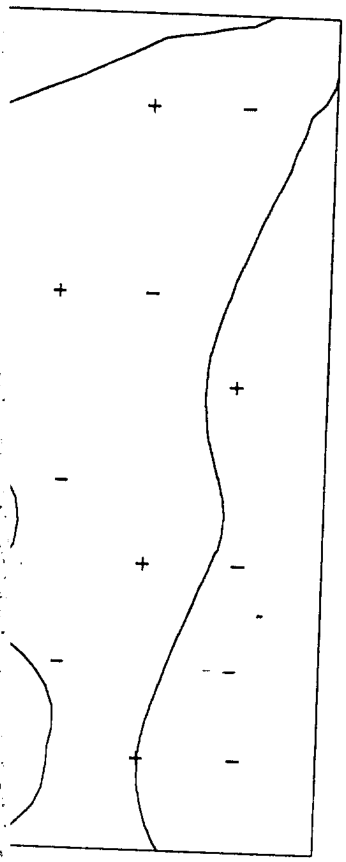

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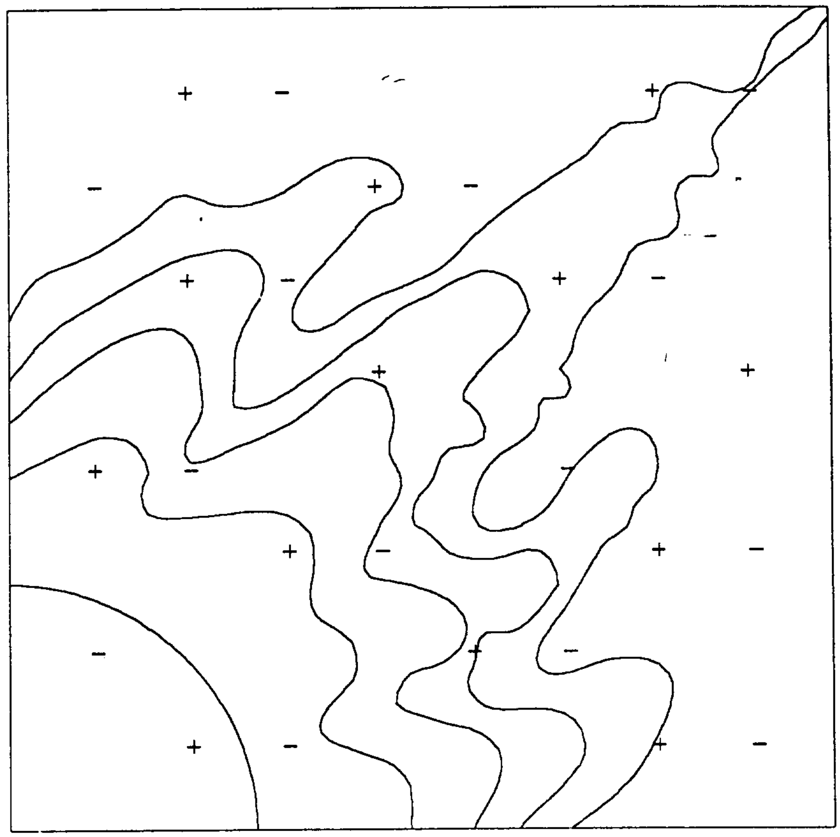

Fig. 3.2 Growth of fingers for immiscible displacement in a heterogeneous reservoir: the temporal evolution of the oil-water discontinuity. The + (−) signs correspond to local maxima (minima) of the permeability field. The viscosity ratio of the water to oil is 1 : 10 .