Math 482 Final Paper: Cantor Spaces in $\mathbb{R}$

This paper describes some basic properties of Cantor subspaces of the real line. It also includes an application of these Cantor subspaces to a characterization of the countability of closed subsets of $\mathbb{R}$ in terms of some simple exterior measures.

Recall that a perfect set is a set for which every point is a limit point. A set $S$ is called totally disconnected if for every $x, y \in S$, there exist disjoint open sets $U, V \subset S$ such that $x \in U$, $y \in V$, and $U \cup V = S$.

**Definition 1.** A Cantor space is a non-empty, totally disconnected, perfect, compact metric space.

**Example 1.** Let $C_0 := [0, 1]$, $C_1 := [0, 1/3] \cup [2/3, 1]$, and $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Similarly, for $i > 2$, let $C_i$ be the closed set given by removing the open middle third of each interval of $C_{i-1}$. The ternary Cantor set

$$\Delta := \bigcap_{i=0}^{\infty} C_i$$

is a Cantor space.

**Proof.** Since $0 \in C_i$ for all $i$, $\Delta$ is non-empty. Since each interval in $C_i$ is of length $3^{-i}$, $\Delta$ is totally disconnected. It is closed and bounded, so compact by the Heine-Borel theorem.

To see that $\Delta$ is perfect, first note that the endpoints of any interval in any $C_i$ remain endpoints of intervals in $C_{i+1}$, and $C_{i+1} \subset C_i$. Hence, every point that is an endpoint of an interval in some $C_i$ is in $\Delta$. Now, fix $x \in \Delta$. Given $\epsilon > 0$, there exists a $C_i$ whose intervals are of length less than $\epsilon$. Hence, both endpoints of the interval in $C_i$ containing $x$ are within $\epsilon$ of $x$, and are members of $\Delta$. Thus, $x$ is a limit point, so $\Delta$ is perfect.

**Theorem 1.** Let $K$ be a Cantor space. If $A \subset K$ is nonempty and clopen, then $A$ is Cantor.

**Proof.** $A$ is compact since it is closed in $K$, and totally disconnected since it is open. To see that $A$ is perfect, let $x \in A$. Since $K$ is perfect, there exists a sequence $(x_n) \subset K$ such that $x_n \to x$. Since $A$ is open, all but a finite number of $x_n$ lie in $A$.

**Theorem 2.** If $A \subset \mathbb{R}$ is a Cantor space, then there is an order-preserving homeomorphism $f : A \to \{0, 1\}^\mathbb{N}$, where $\{0, 1\}^\mathbb{N}$ is ordered lexicographically and equipped with the product metric $d(x, y) = \sum_{i=1}^{n} |x(i) - y(i)| 2^{-i}$.
Theorem 1. Moreover, diam($M$) ≤ $\frac{3}{4}$diam$(A)$ for $i = 0, 1$.

Step 2. For $n > 1$, apply Step 1 to $M_i$ for each $t \in \{0, 1\}^{n-1}$ to get clopen Cantor spaces $M_{t,0}, M_{t,1} \subseteq M_t$ with $M_{t,0} < M_{t,1}$ and diam$(M_{t,i}) \leq \frac{3}{4}$diam$(M_i)$ for $i = 0, 1$. By recursion on $n$, for all $r, s \in \{0, 1\}^n$ we have diam$(M_s) \leq (\frac{3}{4})^n$ diam$(A)$, and if $r < s$ in the lexicographical ordering then $M_r < M_s$, i.e. $x \in M_r, y \in M_s$ implies $x < y$. Moreover, for any fixed $n$, $A = \bigcup_{t \in \{0, 1\}^n} M_t$.

Step 3. Fix $x \in A$. The construction in Step 2 generates a descending sequence of sets $(M_{t_n})_{t_n \in \{0, 1\}^n}$, containing $x$. Since for all $n$ we have $t_{n+1} = t_n$, for some $i \in \{0, 1\}$, this sequence of sets determines a unique element $f(x) = \sum_{i=0}^{\infty} x(i)2^{-i}$ such that, for any $n$, the first $n$ entries of $f(x)$ are $t_n$. To see that $f$ is bijective, note that if $t \in \{0, 1\}^n$ and $t_n = (t(1), t(2), \ldots, t(n))$, then $f^{-1}(t) = \bigcap_{i=0}^{\infty} M_{t_i}$ contains exactly one point, since $M_{t_i}$ is a descending chain of compact sets with diameters going to 0.

To see that $f$ is continuous, let $x \in A$. If $x_m \rightarrow x$ then, for every $M_{t_n}$ containing $x$, all but finitely many $x_m$ lie in $M_{t_n}$, since $M_{t_n}$ is open relative to $A$. Thus, $f(x_m) \rightarrow f(x)$ since diam$(f(M_{t_n})) = 2^{-n} \rightarrow 0$ as $n \rightarrow \infty$. Since $A$ is compact, the continuity of $f$ implies $f^{-1}$ is also continuous.

To see that $f$ is order-preserving, if $x < y$ there exists $n$ so large that $x \in M_n, y \in M_s$ for $s, t$ of length $n$ with $s \neq t$. By Step 2, this implies $s < t$. Hence, $f(x) < f(t)$.

Theorem 3. If $S \subseteq \mathbb{R}$ is a Cantor space, there exists a nondecreasing, onto, continuous function $g : S \rightarrow [0, 1]$.

Proof. Let $h : \{0, 1\}^N \rightarrow [0, 1]$ be defined by $h(x) = \sum_{i=0}^{\infty} x(i)2^{-i}$. Defining $f$ as in Theorem 2, let $g = h \circ f$. Thus, it suffices to show that $h$ is nondecreasing, onto, and continuous.

Let $x, y \in \{0, 1\}^N$. Then $|h(x) - h(y)| = \sum_{i=0}^{\infty} (x(i) - y(i))2^{-i} = \sum_{i=0}^{\infty} |x(i) - y(i)|2^{-i} = d(x, y)$, so $h$ is continuous. If $x < y$, then there exists a minimal $n$ such that $x(n) \neq y(n)$. By the definition of lexicographical ordering, $x(n) = 0$ and $y(n) = 1$. Thus, $h(y) - h(x) = \sum_{i=0}^{\infty} (y(i) - x(i))2^{-i} = 2^{-n} + \sum_{i=n+1}^{\infty} (y(i) - x(i))2^{-i} \geq 2^{-n} + \sum_{i=n+1}^{\infty} (-1)2^{-i} = 0$. Hence, $h$ is nondecreasing. To see that $h$ is onto, let $E_n := \{x \in \{0, 1\}^N : x(i) = 0 \text{ for all } i > n\}$. Then each $h(E_n)$ is a $2^{-n-1}$-net for $[0, 1]$, so the image of $h$ is dense in $[0, 1]$. Since $S$ is compact, $h(S)$ is compact, so $h$ is onto.

Lemma 1. If $f : [a, b] \rightarrow [0, 1]$ is nondecreasing and onto, then $f$ is continuous.

Proof. Let $c \in (a, b]$. Since $f$ is nondecreasing, $\sup_{x < c} f(x) \leq f(c) = \inf_{x \geq c} f(x)$. Hence, since $f$ is onto, $\sup_{x < c} f(x) = f(c)$. To see that $f(c-) = f(c)$, set $\epsilon > 0$. By the definition of supremum, there exists $a < c$ such $f(c) - f(a) < \epsilon$. Then if $a < x < c$, since $f$ is nondecreasing, $f(c) - f(x) < \epsilon$. Hence, $f(c-) = f(c)$.

The proof for right continuity is analogous.\hfill \Box
Lemma 2. Every compact metric space \( K \) can be written as \( K = A \cup B \), where \( A \) is perfect (hence compact), \( B \) is countable, and \( A \cap B = \emptyset \).

Proof. Let \( U \) be a countable base for \( K \). Let \( V := \{ S \in U : S \) is countable\}, and \( W := U \setminus V \). Then \( B := \bigcup_{S \in V} S \) is countable and open. Let \( A := K \setminus B \). Then \( A \) is closed, hence compact.

I claim that \( W := \{ S \cap A : S \in W \} \) is a base for the topology of \( A \). Suppose \( C \subset A \) is open in \( A \), and \( x \in C \). Then \( C \cup B \) is open in \( K \), so there exists \( S \in U \) with \( x \in S \subset (C \cup B) \). Since \( x \notin B \), \( S \) cannot be countable, so \( S \in W \). Hence, \( x \in S \cap A \subset C \), so \( W \) is a base for \( A \).

Note that every element of \( W \) is uncountable, so, since \( B \) is countable, every element of \( W \) is also uncountable. Thus, \( A \) has no isolated points, so \( A \) is perfect.

Definition 2. Given a nondecreasing function \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \), the \( \alpha \)-exterior measure of a set \( E \subset \mathbb{R} \) is defined to be

\[
m^*_\alpha(E) := \inf \left\{ \sum_{i=1}^{\infty} \alpha(b_i) - \alpha(a_i) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}
\]

Theorem 4. Let \( E \subset \mathbb{R} \) be a closed set. Then \( E \) is countable iff \( m^*_\alpha(E) = 0 \) for all nondecreasing, continuous \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \).

Proof. The forward implication is obvious. For the converse, suppose \( E \) were uncountable. If \( E \) contains a nontrivial interval, then let \( \alpha \) be the identity. Since \( E \) contains an interval, it contains a compact set of the form \([a, b]\) for \( a < b \). Hence, any cover of \( E \) by open intervals must contain a finite subcover of \([a, b]\). The sum of the lengths of intervals in this subcover must be at least \( b - a \), so \( m^*_\alpha(E) \geq b - a > 0 \), a contradiction.

Suppose \( E \) does not contain any nontrivial intervals. Note that \( E \cap [n, n+1] \) must be uncountable for some \( n \), so WLOG, \( E \) is compact. Then, by Lemma 2, \( E = A \cup B \) where \( A \) is a Cantor space and \( B \) is countable. Since \( A \subset E \), \( m^*_\alpha(A) \leq m^*_\alpha(E) \), so it suffices to show that \( m^*_\alpha(A) > 0 \).

Let \( f : A \rightarrow [0, 1] \) be the increasing, onto, continuous function defined in Theorem 3. Define

\[
\alpha(x) = \begin{cases} 
0 & : x \leq \inf(A) \\
\sup\{ f(y) : y \in A \cap (-\infty, x) \} & : x > \inf(A)
\end{cases}
\]

Since \( A \) is closed and \( f \) is onto \([0, 1] \), \( \alpha \) is onto \([0, 1] \). Also, \( \alpha \) is clearly non-decreasing. Since \( \alpha \) is constant outside \((\inf(A), \sup(A))\), Lemma 1 implies \( \alpha \) is continuous.

Let \( U \) be a cover of \( A \) by open intervals. Since \( A \) is compact, there exists a finite subcover \( F \subset U \). Denote the elements of \( F \) by \( ((a_i, b_i))_{i=1}^{n} \), sorted so that \( a_i \leq a_{i+1} \) for all \( i < n \). If \( b_{i+1} < b_i \) for some \( i < n \), then \((a_{i+1}, b_{i+1}) \subset (a_i, b_i)\). Since \( F \) is finite, we can recursively throw out all such redundant sets. This procedure only reduces the sum of interval lengths of \( F \), so we may assume
\( b_i \leq b_{i+1} \) for all \( i < n \). For \( i < n \), if \( b_i \geq a_{i+1} \), then \( \alpha(b_i) - \alpha(a_{i+1}) \geq 0 \) since \( \alpha \) is nondecreasing. On the other hand, if \( b_i < a_{i+1} \), then \( \alpha(b_i) - \alpha(a_{i+1}) = 0 \) since \( A \cap [b_i, a_{i+1}] = \emptyset \).

Thus, \( \sum_{i=1}^{n} \alpha(b_i) - \alpha(a_i) \geq \alpha(b_n) - \alpha(a_1) = 1 \). Hence, \( m^*_\alpha(A) \geq 1 \). \( \Box \)

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**References**
