First Exam

Math 304-504, Spring 2015

16 February 2015

1. This exam has 9 questions and 10 pages. Make sure you have all pages before you begin. The ninth question is bonus.

2. This is a 75 minute exam.

3. Put your name and UIN on the front of the exam. Be sure that the exam is stapled together when you turn it in. (See question 1.)

4. This is a closed-book exam. All calculators are prohibited.

Name: ________________________________________________________________

UIN: _________________________________________________________________

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1. (1 point) Make sure your name and UIN are on the front of the exam. Make sure that the exam is stapled together when it is turned in.

2. Indicate whether each of the following statements is true (T) or false (F). (If your handwriting makes it difficult to determine whether you have written a “T” or an “F”, please write out the full word.)

(a) (3 points) If an \( n \times n \) matrix \( A \) is nonsingular, then its reduced row echelon form is the identity matrix \( I \).

(b) (3 points) If \( A \) and \( B \) are both \( n \times n \) matrices, then \( \det(A + B) = \det A + \det B \).

(c) (3 points) If \( A \) is nonsingular then it can be written as the product of elementary matrices.

(d) (3 points) If \( A \) is a triangular \( n \times n \) matrix, then \( \det A \) is the product of the diagonal entries of \( A \).

(e) (3 points) If \( A \) is an \( n \times n \) matrix with \( \det A = 0 \), then the homogeneous equation \( A\mathbf{x} = 0 \) will have nontrivial solutions.
3. Consider the following $3 \times 3$ matrix:

$$A = \begin{pmatrix} x & 3 & 0 \\ 3 & x & 4 \\ 0 & 4 & x \end{pmatrix}$$

(a) (10 points) Find $\det A$. (Your answer should be a function of $x$.)

(b) (5 points) For which values of $x$ is the matrix $A$ singular? Explain.
4. For each of the following matrices, determine whether the matrix is in reduced row echelon form (Y) or not (N).

(a) (3 points) \[
\begin{pmatrix}
1 & 0 & -2 & 0 & 6 \\
0 & 1 & 7 & 0 & 1 \\
0 & 0 & 0 & 1 & 5
\end{pmatrix}
\]

(b) (3 points) \[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(c) (3 points) \[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

(a) _________

(b) _________

(c) _________
5. (15 points) Consider the following matrix:

\[
A = \begin{pmatrix}
1 & 0 & 3 \\
1 & 1 & -1 \\
4 & 2 & 3
\end{pmatrix}
\]

Is the matrix \( A \) singular? If so, how do you know? If not, find its inverse.
6. (20 points) Find all solutions of the following system of equations:

\begin{align*}
    x_1 + 2x_2 + x_3 + x_4 &= 2 \\
    x_1 + 3x_2 + 2x_3 + 2x_4 &= 3 \\
    2x_1 + 3x_2 + x_3 + 2x_4 &= 4 \\
    x_1 - x_3 + x_4 &= 2
\end{align*}
7. Suppose an $m \times n$ matrix $A$ is given in terms of its columns (here $m \geq 3$):

$$A = \begin{pmatrix} | & | & \cdots & | \\ a_1 & a_2 & \ldots & a_3 \\ | & | & \cdots & | \end{pmatrix}$$

Suppose further that the first three columns of $A$ satisfy

$$a_1 + a_2 + a_3 = 0.$$ 

Consider now the homogeneous equation $Ax = 0$.

(a) (8 points) Show that the vector

$$x = (1, 1, 1, 0, \ldots, 0)^T$$

satisfies $Ax = 0$.

(b) (7 points) Is the matrix $A$ singular? Explain why or why not.
8. (10 points) Use *Cramer’s rule* to find the solution(s) of the following $2 \times 2$ system of linear equations. (Note: AT MOST half-credit will be given for using a different method.)

\[
\begin{align*}
2x_1 - x_2 &= 1 \\
x_1 + 2x_2 &= 3
\end{align*}
\]
9. (This question is longer and is worth less than other questions so you are advised not to work on it unless you have finished the rest of the exam. I also reserve the right to be sneaky with bonus questions.) Given a vector \( x \in \mathbb{R}^{n+1} \), the \((n+1) \times (n+1)\) matrix \( V \) defined by

\[
    v_{ij} = \begin{cases} 
        1 & \text{if } j = 1 \\
        x_{i}^{j-1} & \text{for } j = 2, \ldots, n + 1
    \end{cases}
\]

is called the Vandermonde matrix.

(a) (3 points (bonus)) Show that if \( Vc = y \) and

\[
    p(x) = c_1 + c_2 x + \cdots + c_{n+1} x^n,
\]

then

\[
    p(x_i) = y_i, \quad i = 1, 2, \ldots, n + 1
\]

(b) (4 points (bonus)) Write down the Vandermonde matrix for \( n = 2 \) and compute its determinant. What conditions must the entries of \( x \) satisfy for \( V \) to be nonsingular?
(c) (3 points (bonus)) Now consider it again for general $n$. Suppose that $x_1, x_2, \ldots, x_{n+1}$ are all distinct. Show that if $c$ is a solution of $Vc = 0$, then all coefficients $c_1, c_2, \ldots, c_{n+1}$ must be zero.