1. This exam has 8 questions and 12 pages. Make sure you have all pages before you begin. The eighth question is bonus (and worth less than the others).

2. This is a 75 minute exam.

3. Put your name and UIN on the front of the exam. Be sure that the exam is stapled together when you turn it in. (See question 1.)

4. This is a closed-book exam. All calculators are prohibited.

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UIN: 

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1. (2 points) Make sure your name and UIN are on the front of the exam. Make sure that
the exam is stapled together when it is turned in.

2. Indicate whether each of the following statements is true (T) or false (F). (If your
handwriting makes it difficult to determine whether you have written a “T” or an “F”,
please write out the full word.)

(a) (3 points) If $A$ is an $m \times n$ matrix then $A$ and $A^\top$ have the same rank.

(a) [ ] T

(b) (3 points) If $U$ is the reduced row echelon form of a matrix $A$, then $A$ and $U$
have the same column space.

(b) [ ] F

(c) (3 points) Suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation. If
$L(x_1) = L(x_2)$ then $x_1 = x_2$.

(c) [ ] F

(d) (3 points) If $L : V \to V$ is a linear transformation and $x \in \ker(L)$, then
$L(v + x) = L(v)$ for all $v \in V$.

(d) [ ] T

(e) (3 points) If $S$ and $T$ are subspaces of a vector space $V$, then $S \cap T$ is also a subspace
of $V$. (Here $S \cap T = \{v \in V : v \in S$ and $v \in T\}$.)

(e) [ ] T
3. For each of the following questions, write the answer (a number) in the allotted space.

(a) (4 points) Suppose $B$ is a $2 \times 5$ matrix and suppose that $C(B)$ (the column space of $B$) is 2-dimensional. What is the dimension of $R(B)$ (the row space of $B$)?

(a) 2

(b) (4 points) Suppose $D$ is a $2 \times 5$ matrix and suppose that $C(D)$ (the column space of $D$) is 2-dimensional. What is the dimension of $N(D)$ (the null space of $D$)?

(b) 3
4. (15 points) Consider the following matrix:

$$ A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -7 \end{pmatrix} $$

Find $N(A)$ (the null space of $A$).

**Solution:** We row reduce $A$:

$$ A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -8 \\ 0 & -1 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & 9 \end{pmatrix} $$

We then have that the null space is cut out by the two equations:

$$ x_1 - 8x_3 = 0 $$
$$ x_2 + 9x_3 = 0 $$

There is thus one free variables $x_3$, and the null space is given by

$$ \left\{ \begin{pmatrix} 8 \alpha \\ -9 \alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\} = \text{Span} \left( \begin{pmatrix} 8 \\ -9 \\ 1 \end{pmatrix} \right) $$

so that the following vector forms a basis for $N(A)$:

$$ \left\{ \begin{pmatrix} 8 \\ -9 \\ 1 \end{pmatrix} \right\} $$
5. Consider the following vectors \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) in \( \mathbb{R}^3 \):

\[
\begin{align*}
\mathbf{v}_1 &= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \\
\mathbf{v}_2 &= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \\
\mathbf{v}_3 &= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}
\end{align*}
\]

(a) (10 points) Determine whether \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) are linearly independent.

**Solution:** If we form the matrix \( A \) with these vectors as the columns, it is enough to check that the null space of \( A \) is \{0\}. Because \( A \) is \( 3 \times 3 \), we can check this by verifying that \( \det A \neq 0 \):

\[
\det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 3 \end{pmatrix} = 1(0) - 2(6) + 1(0) = -12 \neq 0
\]

The vectors are thus linearly independent.

(b) (5 points) What is the dimension of \( \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \) (the span of \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \))? Explain your answer.

**Solution:** We have three linearly independent vectors in \( \mathbb{R}^3 \), which is 3-dimensional. The three vectors thus must span all of \( \mathbb{R}^3 \) and so the dimension of their span is 3.
6. Consider the subset \( S \subset \mathbb{R}^4 \) given by

\[
S = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 + x_2 + x_3 + x_4 = 0 \right\}.
\]

(a) (10 points) Show that \( S \) is a subspace of \( \mathbb{R}^4 \).

**Solution:** We must check that it is nonempty, closed under addition, and closed under scalar multiplication.

- \( \mathbf{0} \in S \) because \( 0 + 0 + 0 + 0 = 0 \), so \( S \) is not empty.
- Suppose \( \mathbf{x}, \mathbf{y} \in S \). Then

\[
\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{pmatrix}
\]

and we have \( (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) + (x_4 + y_4) = (x_1 + x_2 + x_3 + x_4) + (y_1 + y_2 + y_3 + y_4) = 0 + 0 = 0 \), so \( \mathbf{x} + \mathbf{y} \in S \) and \( S \) is closed under scalar multiplication.

- Suppose that \( \mathbf{x} \in S \) and \( \alpha \) is a scalar. We then have that \( \alpha x_1 + \alpha x_2 + \alpha x_3 + \alpha x_4 = \alpha(x_1 + x_2 + x_3 + x_4) = 0 \), so \( \alpha \mathbf{x} \in S \).

Thus \( S \) is a subspace.

(b) (2 points) Show that \( S \neq \mathbb{R}^4 \).

**Solution:** The vector \( \mathbf{e}_1 \in \mathbb{R}^4 \) is not in \( S \), so \( S \neq \mathbb{R}^4 \).
(c) (10 points) Show that

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & -1 & -1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 1 \\
-1 & -1 & -1
\end{pmatrix}
\]

is a basis for \( S \).

Solution: First note that all three vectors are in \( S \). Note also that the dimension of \( S \) is at most 3 because \( S \neq \mathbb{R}^4 \). We now check that the three vectors are linearly independent by putting them as columns of a matrix and examining its null space:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & -1 & -1
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

which has null space \( \{0\} \), so the three vectors are linearly independent and span a three dimensional subspace of \( S \). Because \( \dim S \leq 3 \), they must span all of \( S \) and thus be a basis.

(d) (3 points) Find the dimension of \( S \).

Solution: We found a basis with three elements, so \( \dim S = 3 \).
7. Let \( T \) be the following matrix:

\[
T = \begin{pmatrix}
1 & 1 \\
2 & 3
\end{pmatrix}
\]

(a) (10 points) Find \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) so that \( T \) is the transition matrix from the basis \( F = \{\mathbf{u}_1, \mathbf{u}_2\} \) to the standard basis \( E = \{\mathbf{e}_1, \mathbf{e}_2\} \) of \( \mathbb{R}^2 \).

**Solution:** The first column corresponds to \( \mathbf{u}_1 \) expressed in terms of the standard basis and the second column to \( \mathbf{u}_2 \), so

\[
\mathbf{u}_1 = \begin{pmatrix}
1 \\
2
\end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix}
1 \\
3
\end{pmatrix}
\]
(b) (10 points) Consider the linear transformation \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by
\[
L(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) = \alpha \mathbf{u}_1 + 2\beta \mathbf{u}_2.
\]
(Here \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) are the vectors you found in the previous part.) Write down the matrix representing \( L \) with respect to the standard basis. (Hint: you do not actually need to know what \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) are to do this part.)

**Solution:** You can either compute this directly or you can use similarity. The matrix \( B \) representing \( L \) with respect to the basis \( E \) is given by
\[
B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}
\]
The transition matrix corresponding to basis change from \( F \) to \( E \) is given by \( T \) above and the transition matrix corresponding to change from \( E \) to \( F \) is given by its inverse:
\[
T^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}
\]
The matrix representing \( L \) with respect to \( E \) is thus given by \( TBT^{-1} \), which we calculate:
\[
A = TBT^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -6 & 4 \end{pmatrix}
\]
8. For this problem, we let $V = P_3$ be the vector space of polynomials of degree at most 2 and we define, for $p \in V$,

$$T(p(x)) = (x - 1)p'(x) - p(x).$$

(a) (1 point (bonus)) Show that if $p \in V$, then $T(p) \in V$.

**Solution:** If $p \in V$, then $\deg p' \leq 1$, so $\deg T(p) \leq \max(1 + 1, 2) = 2$, so $T(p) \in V$.

(b) (3 points (bonus)) Show that $T$ is a linear transformation.

**Solution:** We must check that $T$ respects addition and scalar multiplication. Suppose $p, q \in V$ and $\alpha, \beta$ are scalars. Then

$$T(\alpha p + \beta q)(x) = (x - 1) [\alpha p'(x) + \beta q'(x)] - (\alpha p(x) + \beta q(x))$$

$$= \alpha(x - 1)p'(x) - \alpha p(x) + \beta(x - 1)q'(x) - \beta q(x)$$

$$= \alpha T(p(x)) + \beta T(q(x)),$$

so $T$ is a linear transformation.
(c) (2 points (bonus)) Consider the (ordered) basis $E = \{1, x, x^2\}$ of $V$. Find the matrix $A$ representing $T$ with respect to this basis.

**Solution:** We calculate the action of $T$ on the basis elements:

- $T(1) = -1 + 0x + 0x^2$
- $T(x) = (x - 1) - x = -1 + 0x + 0x^2$
- $T(x^2) = (x - 1)(2x) - x^2 = 0 - 2x + x^2$

so that the matrix $A$ is given by

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) (1 point (bonus)) Consider the (ordered) basis $F = \{1, x-1, (x-1)^2\}$. Write down the matrix $B$ representing $T$ with respect to the basis $E$.

**Solution:** We calculate the action of $T$ on the basis vectors:

- $T(1) = -1 + 0(x - 1) + 0(x - 1)^2$
- $T(x - 1) = (x - 1) - (x - 1) = 0 + 0(x - 1) + 0(x - 1)^2$
- $T((x - 1)^2) = (x - 1)(2(x - 1)) - (x - 1)^2 = 0 + 0(x - 1) + (x - 1)^2$

so that the matrix $B$ is given by the following:

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
(e) (3 points (bonus)) Show that the matrices $A$ and $B$ that you found in the previous two parts are similar. (In other words, find a matrix $S$ so that $A = SBS^{-1}$.)

**Solution:** The matrix $A$ represents $T$ with respect to the basis $E$, while $B$ represents $T$ with respect to the basis $F$. Thus if $S$ is the transition matrix corresponding to the change from $F$ to $E$, then we will have $A = SBS^{-1}$.

We express the basis $F$ in terms of the basis $E$:

\[
1 = 1 + 0x + 0x^2 \\
x - 1 = -1 + 1x + 0x^2 \\
(x - 1)^2 = 1 - 2x + x^2
\]

so the transition matrix corresponding to the change from $F$ to $E$ is given by

\[
S = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}
\]

The transition matrix corresponding to the change from $E$ to $F$ is then given by

\[
S^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}
\]

We verify this:

\[
SBS^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} = A
\]