Second Exam
Math 304-504, Spring 2015
30 March 2015

1. This exam has 8 questions and 12 pages. Make sure you have all pages before you begin. The eighth question is bonus (and worth less than the others).

2. This is a 75 minute exam.

3. Put your name and UIN on the front of the exam. Be sure that the exam is stapled together when you turn it in. (See question 1.)

4. This is a closed-book exam. All calculators are prohibited.

Name: ____________________________________________

UIN: ______________________________________________

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1. (2 points) Make sure your name and UIN are on the front of the exam. Make sure that the exam is stapled together when it is turned in.

2. Indicate whether each of the following statements is true (T) or false (F). (If your handwriting makes it difficult to determine whether you have written a “T” or an “F”, please write out the full word.)

(a) (3 points) If $A$ is an $m \times n$ matrix then $A$ and $A^T$ have the same rank.

(b) (3 points) If $U$ is the reduced row echelon form of a matrix $A$, then $A$ and $U$ have the same column space.

(c) (3 points) Suppose $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transformation. If $L(x_1) = L(x_2)$ then $x_1 = x_2$.

(d) (3 points) If $L : V \rightarrow V$ is a linear transformation and $x \in \ker(L)$, then $L(v + x) = L(v)$ for all $v \in V$.

(e) (3 points) If $S$ and $T$ are subspaces of a vector space $V$, then $S \cap T$ is also a subspace of $V$. (Here $S \cap T = \{ v \in V : v \in S \text{ and } v \in T \}$.)

(a) __________

(b) __________

(c) __________

(d) __________

(e) __________
3. For each of the following questions, write the answer (a number) in the allotted space.

(a) (4 points) Suppose $B$ is a $2 \times 5$ matrix and suppose that $C(B)$ (the column space of $B$) is 2-dimensional. What is the dimension of $R(B)$ (the row space of $B$)?

(a) __________

(b) (4 points) Suppose $D$ is a $2 \times 5$ matrix and suppose that $C(D)$ (the column space of $D$) is 2-dimensional. What is the dimension of $N(D)$ (the null space of $D$)?

(b) __________
4. (15 points) Consider the following matrix:

\[ A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -7 \end{pmatrix} \]

Find \( N(A) \) (the null space of \( A \)).
5. Consider the following vectors \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) in \( \mathbb{R}^3 \):

\[
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}
\]

(a) (10 points) Determine whether \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) are linearly independent.

(b) (5 points) What is the dimension of \( \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \) (the span of \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \))? Explain your answer.
6. Consider the subset $S \subset \mathbb{R}^4$ given by

$$S = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 + x_2 + x_3 + x_4 = 0 \right\}.$$

(a) (10 points) Show that $S$ is a subspace of $\mathbb{R}^4$.

(b) (2 points) Show that $S \neq \mathbb{R}^4$. 
(c) (10 points) Show that

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & -1
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 \\
0 & 0 \\
-1 & -1
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 \\
0 & 1 \\
-1 & -1
\end{bmatrix}
\]

is a basis for \( S \).

(d) (3 points) Find the dimension of \( S \).
7. Let $T$ be the following matrix:

$$T = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

(a) (10 points) Find $u_1$ and $u_2$ so that $T$ is the transition matrix from the basis $F = \{u_1, u_2\}$ to the standard basis $E = \{e_1, e_2\}$ of $\mathbb{R}^2$. 
(b) (10 points) Consider the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$L(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) = \alpha \mathbf{u}_1 + 2\beta \mathbf{u}_2.$$ 

(Here $\mathbf{u}_1$ and $\mathbf{u}_2$ are the vectors you found in the previous part.) Write down the matrix representing $L$ with respect to the standard basis. (Hint: you do not actually need to know what $\mathbf{u}_1$ and $\mathbf{u}_2$ are to do this part.)
8. For this problem, we let $V = P_3$ be the vector space of polynomials of degree at most 2 and we define, for $p \in V$,

$$T(p(x)) = (x - 1)p'(x) - p(x).$$

(a) (1 point (bonus)) Show that if $p \in V$, then $T(p) \in V$.

(b) (3 points (bonus)) Show that $T$ is a linear transformation.
(c) (2 points (bonus)) Consider the (ordered) basis $E = \{1, x, x^2\}$ of $V$. Find the matrix $A$ representing $T$ with respect to this basis.

(d) (1 point (bonus)) Consider the (ordered) basis $F = \{1, x-1, (x-1)^2\}$. Write down the matrix $B$ representing $T$ with respect to the basis $E$. 
(e) (3 points (bonus)) Show that the matrices $A$ and $B$ that you found in the previous two parts are similar. (In other words, find a matrix $S$ so that $A = SBS^{-1}$.)